BRNO UNIVERSITY OF TECHNOLOGY

VYSOKÉ UČENÍ TECHNICKÉ V BRNĚ

FACULTY OF ELECTRICAL ENGINEERING AND COMMUNICATION

FAKULTA ELEKTROTECHNIKY
A KOMUNIKAČNÍCH TECHNOLOGIÍ

DEPARTMENT OF MATHEMATICS

ÚSTAV MATEMATIKY

OPTIMIZATION OF DELAYED DIFFERENTIAL SYSTEMS BY LYAPUNOV'S DIRECT METHOD

OPTIMALIZACE DIFERENCIÁLNÍCH SYSTÉMŮ SE ZPOŽDĚNÍM UŽITÍM PŘÍMÉ METODY LYAPUNOVA

SHORT VERSION OF PHD THESIS

ZKRÁCENÁ VERZE DIZERTAČNÍ PRÁCE

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BRNO 2018

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Optimální stabilizace, řídící funkce, funkcionál Lyapunova-Krasovského, asymptotická stabilita, Malkinova metoda

KEYWORDS

Optimal stabilization, control function, Lyapunov-Krasovskii functional, asymptotic stability, Malkin's approach

MÍSTO ULOŽENÍ PRÁCE

Vědecké oddělení, Fakulta elektrotechniky a komunikačních technologií, Vysoké učení technické v Brně, Technická 3058/10, 616 00 Brno

ISBN 978-80-214-XXXX
ISSN 1213-418X

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1 INTRODUCTION

Differential equations are a strong tool for modelling and solving numerous engineering, mechanical, economic or population problems. It is well-known that in such problems a time delay arises quite naturally. For example, in electrical engineering, the time delay can be measured as the difference between the input of a signal in an electrical circuit and its response. In general, there is always a time delay in the real-life processes depending on time. So, differential equations with time delay are an important field of research. As the systems with feedback can be described (under certain conditions) by systems of differential equations with a delay or by difference equations, a wide range of applications is opened for research.

In practical applications, the behaviour of many dynamical systems depends on their previous history. This phenomenon can be brought about by the presence of delays in the equations under consideration. In view of the intrinsic difficulties in solving such problems, progress in this field is slow. This is why using the optimal control of delay systems is so needed and important.

The thesis is devoted to the optimal control problem of delayed differential equations.

The fundamentals of the theory of functional and ordinary differential equations are well described, for example, in books by R.D. Driver [12], J.K. Hale [22], L.E. Elsgolts and S.B. Norkin [13], N.N. Krasovskii [25], R.P. Agarwal, L. Berezansky, E. Braverman and A. Domoshnitsky [3], R.P. Agarwal, M. Bohner and Li Wan-Tong [4], I. Gyori and G. Ladas [21]. Classics in the field of the optimal control are R. Bellman [9], L.S. Pontryagin, V.G. Boltyanskij, R.V. Gamkrelidze and E.F. Mishchenko [40], A.A. Fel'dbaum [14], A.M. Letov [28], [29], V.M. Alekseev, V.M. Tikhomirov and S.V. Fomin [5], I.G. Malkin [33], R. Gabasov and F. Kirillova [18], [19], D.E. Kirk [24], E. Fridman [16], A.V. Kim and A.V. Ivanov [23].

1.1 CURRENT STATE

Differential equations have been the object of research since the 17th century (after works by Newton and Leibniz) and have been intensively developed for the last century. Monographs summarizing some outcomes were mentioned above in the Introduction.

One of the most important sections of the qualitative theory of functional differential equations is the theory of stability. The method of Lyapunov functionals, proposed by Krasovskii in [25], is still one of the main methods in the research of the delayed system's stability. Analytical research of the stability of some dynamic systems led to the emergence of a new independent field of science - the theory of automatic control (regulation). The basis of this theory is usually associated with the book [35]. The theory of optimal control is an important part in the theory of automatic control, formed primarily on the basis of the classical calculus of variations, the Pontryagin maximum principle [40] and Bellman's dynamic programming [9]. The direct Lyapunov method (Lyapunov function method) [31], which is the basis of the modern nonlinear theory of automatic control, is widely used in modeling control structures of nonlinear systems. In addition to the stability conditions, the method includes an analysis of the quality of control processes.

Numerous papers on the qualitative theory of differential equations, control theory, and optimization are published every year. Some interesting results have been published on representations of solutions of delayed systems [11], [26], on stability of solutions [17], [30], and on optimal control for delayed differential equations [36], [42], [45], [49]. Functional differential equations for modeling the biological problems were first used and investigated in [44]. There are many later works on modeling of biological processes, for example, [1], [2], [20], on applying optimal control in biology and medicine [7], [27], [41]. In [47], the authors introduced a version of the stochastic discrete-time maximum principle for solving an optimal control problem. In [37], the damping of the solution problem is solved by means of a linear difference–differential controller with a state feedback. Here a certain form of the control function was used to stabilize the solution. There are numerous works (for example, [15], [46]) where the authors study the control of systems using some specific control functions.

1.2 AIMS OF THE THESIS

The aim of the thesis is to solve the optimal stabilization problem for processes described by a system of delayed differential equations

$$x'(t) = f(t, x_t, u), \quad t \geq t_0,$$

where $t_0 \in \mathbb{R}$, f is defined on a subspace of $[t_0, \infty) \times C_\tau^m \times \mathbb{R}^r$, $m, r \in \mathbb{N}$, $C_\tau^m = C([-\tau, 0], \mathbb{R}^m)$, $\tau > 0$, $x_t(\theta) := x(t+\theta)$, $\theta \in [-\tau, 0]$, $x\colon [t_0 - \tau, \infty) \to \mathbb{R}^m$. Under the assumption $f(t, \theta_m^*, \theta_r) = \theta_m$, where $\theta_m^* \in C_\tau^m$ is a zero vector-function, θ_r and θ_m are r and m-dimensional zero vectors, a control function $u = u(t, x_t)$, $u\colon [t_0, \infty) \times C_\tau^m \to \mathbb{R}^r$, $u(t, \theta_m^*) = \theta_r$ is such that the zero solution $x(t) = \theta_m$, $t \geq t_0 - \tau$ of the system

$$x'(t) = f(t, x_t, u(t, x_t)), \quad t \geq t_0,$$

is asymptotically stable and, for an arbitrary solution $x = x(t)$, the integral

$$\int_{t_0}^{\infty} \omega\left(t, x_t, u(t, x_t)\right) \mathrm{d}t,$$

where ω is a positive-definite functional, exists and attains its minimum value in a given sense. The thesis solves a problem of optimal stabilization for differential non-delayed and delayed equations and their systems.

The motivation of our research goes back to the results by I.G. Malkin. His book [33] (we refer to the original book written in Russian, to the best of our knowledge, there is no translation into English of the second revised edition, the book [34] is an English translation of the first edition of Malkin's book and does not include the results mentioned) contains, among others, a general principle related to optimal stabilization of ordinary differential systems and its application to linear ordinary differential systems. This principle we apply to some types of linear differential equations and their systems to solve optimal control problems. We analyzed Malkin's approach and, as a result of our investigation, we present its modification to differential delayed systems. Illustrative examples showing how this principle can be applied are developed and, in addition, linear differential delayed systems are considered.

Some results of this work have been already published by the author of the thesis, as a co-author, e.g., in [52]– [60].

1.3 PRELIMINARIES

For the auxiliary material given in this part for the reader's convenience, we refer, for example, to [12, 22].

Let $C^m_\tau = C([-\tau, 0], \mathbb{R}^m)$, where $\tau > 0$, $m \in \mathbb{N}$, be the Banach space of continuous mappings $\varphi\colon [-\tau, 0] \to \mathbb{R}^m$. If A is any set in $\mathbb{R}^m$, we will set $C^m_\tau(A) = C([-\tau, 0], A)$.

Let $C^m_\tau(D)$ be the space of continuous mappings from the interval $[-\tau, 0]$ into the set $D = \{\xi \in \mathbb{R}^m \colon \|\xi\| < M\}$, M is a positive constant (or $M = \infty$).

For each $t \geq t_0$, we define $x_t \in C^m_\tau$ by $x_t(\theta) = x(t + \theta)$, $\theta \in [-\tau, 0]$.

Consider a delayed differential system

$$x'(t) = G(t, x_t), \tag{1.1}$$

where $G\colon [\alpha, \infty) \times C^m_\tau(D) \to \mathbb{R}^m$ and $\alpha \in \mathbb{R}$. Given any $t_0 \geq \alpha$ and any $\varphi \in C^m_\tau(D)$, we shall study (1.1) in conjunction with the initial condition

$$x_{t_0} = \varphi. \tag{1.2}$$

Let $x\colon [t_0 - \tau, \infty) \to \mathbb{R}^m$ be a continuous vector-function, $t_0 \in \mathbb{R}$, and let $\tau > 0$ be a number. To emphasize the dependence of x on t_0 and φ, we will sometimes denote $x(t)$ by $x(t; t_0, \varphi)$. Let $\beta_1 \in \mathbb{R}$, $t_0 < \beta_1 \leq \infty$.

Definition 1.3.1. A continuous function $x\colon [t_0 - \tau, \beta_1) \to D$ with $\beta_1 \in (t_0, \infty)$ is called a solution of the initial problem (1.1), (1.2) on $[t_0 - \tau, \beta_1)$ if the equation (1.1) is satisfied on $[t_0, \beta_1)$ and if $x(t_0 + \theta) = \varphi(\theta)$ for every $\theta \in [-\tau, 0]$.

For a given $t \in [t_0, \infty)$, we define a norm

$$\|x(t)\|_\tau := \max_{\theta \in [-\tau, 0]} (\|x(t + \theta)\|),$$

where

$$\|x(s)\| := \max_{i=1,\dots,n} \{|x_i(s)|\},\ s \in [t_0 - \tau, \infty). \tag{1.3}$$

If $\varphi \in C^m_\tau$ then

$$\|\varphi\|_\tau := \max_{\theta \in [-\tau, 0]} \{\|\varphi(\theta)\|\},$$

where

$$\|\varphi(\theta)\| := \max_{i=1,\dots,m} \{|\varphi_i(\theta)|\}.$$

Let us assume that, for each $t_0 \geq \alpha$, G satisfies the following Condition (C) on $[t_0, \infty) \times C^m_\tau(D)$.

Definition 1.3.2. Condition (C) We say that the functional $G(t, x_t)$ is continuous if it is continuous with respect to t in $[t_0, \infty)$ for each given continuous function $x\colon [t_0 - \tau, \infty) \to \mathbb{R}^m$.

If G satisfies Condition (C), then a continuous function $x\colon [t_0 - \tau, \beta_1) \to D$ is a solution of the initial problem (1.1), (1.2) if and only if

$$x(t) = \begin{cases} \varphi(t - t_0) & \text{for} \quad t_0 - \tau \leq t \leq t_0, \\[2ex] \varphi(0) + \displaystyle\int_{t_0}^{t} G(s, x_s)\mathrm{d}s & \text{for} \quad t_0 \leq t < \beta_1. \end{cases}$$

Moreover, we will assume that G is locally Lipschitzian and quasi-bounded, see definitions below. Let the symbol J mean either $[t_0, \infty)$ or $[\alpha, \infty)$ as required.

Definition 1.3.3. The functional $G\colon J \times C_\tau^m(D) \to \mathbb{R}^m$ is locally Lipschitzian if, for each given $(t^*, \varphi^*) \in J \times C_\tau^m(D)$, there exist numbers $a > 0$ and $b > 0$ such that

$$\tilde{C} \equiv ([t^* - a, t^* + a] \cap J) \times \{\varphi \in C_\tau^m : \|\varphi - \varphi^*\|_\tau \le b\}$$

is a subset of $J \times C_\tau^m(D)$ and G is Lipschitzian on $\tilde{C}$. In other words, for some number K (a Lipschitz constant depending on $\tilde{C}$),

$$\|G(t,\varphi) - G(t,\varphi^*)\| \le K\|\varphi - \varphi^*\|_\tau$$

whenever $(t,\varphi) \in \tilde{C}$ and $(t,\varphi^*) \in \tilde{C}$.

Definition 1.3.4. The functional $G\colon [t_0,\infty) \times C_\tau^m(D) \to \mathbb{R}^m$ is said to be quasi-bounded if G is bounded on every set of the form $[t_0,\beta_1] \times C_\tau^m(A)$, where $t_0 < \beta_1 < \infty$ and A is a closed bounded set of D.

The properties described in Definitions 1.3.2–1.3.4 are basic for ensuring, for example, the existence and uniqueness of a noncontinuable solution of the problem (1.1), (1.2), see Theorem 1.3.5 below, and its continuation (Theorem 1.3.7).

Theorem 1.3.5. ***(Local Existence)*** *Let $G\colon [t_0,\infty) \times C_\tau^m(D) \to \mathbb{R}^m$ satisfy Condition (C) and let it be locally Lipschitzian. Then, for each $\varphi \in C_\tau^m(D)$, the initial problem* (1.1), (1.2) *has a unique solution on $[t_0 - \tau, t_0 + \triangle)$ for some $\triangle > 0$.*

Definition 1.3.6. Let x on $[t_0 - \tau, \beta_1)$ and y on $[t_0 - \tau, \beta_2)$, $\beta_2 > t_0$, both be solutions of the initial problem (1.1), (1.2). If $\beta_2 > \beta_1$, we say y is a continuation of x, or x can be continued to $[t_0 - \tau, \beta_2)$. A solution $x(t)$ of the initial problem (1.1), (1.2) is noncontinuable (on an interval $[t_0 - \tau, \infty)$) if it has no continuation.

Theorem 1.3.7. ***(Extended Existence)*** *Let $G\colon [t_0,\infty) \times C_\tau^m(D) \to \mathbb{R}^m$ satisfy Condition (C) and let it be locally Lipschitzian and quasi-bounded. Then, for each $\varphi \in C_\tau^m(D)$, the problem* (1.1), (1.2) *has a unique noncontinuable solution x on $[t_0 - \tau, \beta_1)$; if $\beta_1 < \infty$, then, for every closed bounded set $A \subset D$, $x(t) \notin A$ for some t in (t_0, β_1).*

Definition 1.3.8. The trivial solution of (1.1) is said to be stable at $t_0 \ge \alpha$ (in the sense of Lyapunov) if, for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that, whenever $\|\varphi\|_\tau < \delta$, the solution $x(t; t_0, \varphi)$ exists on $[t_0 - \tau, \infty)$ and

$$\|x(t; t_0, \varphi)\| < \varepsilon$$

for all $t \ge t_0 - \tau$. Otherwise, the trivial solution is said to be unstable at t_0. The trivial solution of (1.1) is said to be uniformly stable on $[\alpha, \infty)$ if it is stable at each $t_0 \ge \alpha$ and the number δ is independent of t_0, i.e., $\delta = \delta(\varepsilon)$ depends only on ε.

Definition 1.3.9. Let $\bar{x}\colon (\alpha - \tau, \infty) \to D$ satisfy the equation (1.1) on $[\alpha, \infty)$. We say that $\bar{x}$ is stable at $t_0 \ge \alpha$ (in the sense of Lyapunov) if, for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that, whenever $\|\varphi - \bar{x}_{t_0}\|_\tau < \delta$, it follows that $x(\cdot; t_0, \varphi)$ exists on $[t_0 - \tau, \infty)$ and

$$\|x(t; t_0, \varphi) - \bar{x}(t)\| < \varepsilon$$

for all $t \ge t_0 - \tau$. Otherwise, the solution $\bar{x}$ is said to be unstable at t_0 (in the sense of Lyapunov). The solution $\bar{x}$ of (1.1) is said to be uniformly stable on $[\alpha, \infty)$ if it is stable at each $t_0 \ge \alpha$ and the number δ is independent of t_0, i.e., $\delta = \delta(\varepsilon)$ depends only on ε.

Definition 1.3.10. The trivial solution of (1.1) is said to be uniformly asymptotically stable if it is uniformly stable and there exists a δ_1 (independent of t_0) such that, whenever $t_0 \geq \alpha$ and $\|\varphi\|_\tau < \delta_1$, the expression

$$x(t; t_0, \varphi)$$

tends to 0 as $t \to \infty$ in the following manner:
For each $\eta > 0$, there exists $T = T(\eta) > 0$ (independent of t_0) such that

$$\|x(t; t_0, \varphi)\| < \eta$$

for all $t \geq t_0 + T$.

The following definitions are related to the estimation of functionals. Throughout the thesis, we will denote by $V = V(t, x_t)$ a functional such that

$$V\colon [t_0, \infty) \times C_\tau^m \to \mathbb{R}. \tag{1.4}$$

Definition 1.3.11. Let a functional V be given. It is called positive-definite if there exists a continuous non-decreasing function $w\colon [0, M) \longrightarrow \mathbb{R}$, $w(0) = 0$, $w(s) > 0$ if $s \in (0, M)$ such that

$$V(t, \psi) \geq w(\|\psi(0)\|)$$

on $(\alpha, \infty) \times C_\tau^m(D)$.

Definition 1.3.12. Let a functional V be given. V is said to have an infinitesimal upper bound if there exists a continuous non-decreasing function $W\colon [0, M) \longrightarrow \mathbb{R}$, $W(0) = 0$, $W(s) > 0$ if $s \in (0, M)$ such that

$$V(t, \psi) \leq W(\|\psi\|_\tau)$$

on $(\alpha, \infty) \times C_\tau^m(D)$.

Definition 1.3.13. A positive-definite functional V having an infinitesimal upper bound is called a Lyapunov-Krasovskii functional.

Definition 1.3.14. Let $x\colon [t_0 - \tau, \infty) \to \mathbb{R}^m$. The derivative of a functional $V(t, x_t)$ at a point $t \geq t_0$ is defined as

$$\frac{\mathrm{d}V(t, x_t)}{\mathrm{d}t} := \lim_{\Delta \to 0} \frac{V(t + \Delta, x_{t+\Delta}) - V(t, x_t)}{\Delta},$$

provided that the limit exists and is finite.

Below we assume that there exists the derivative $\mathrm{d}V(t, x_t)/\mathrm{d}t$ of the functional $V(t, x_t)$ along the trajectories of the differential delayed systems considered, that is, we will assume that x is a solution of a given system.

Theorem 1.3.15. *If there exists a Lyapunov-Krasovskii functional V and if it defines a non-increasing function of t on $[t_0, \infty)$ whenever*

$$x = x(\cdot; t_0, \varphi),\ t \in [t_0 - \tau, \infty)$$

is the noncontinuable solution of (1.1) *through some $(t_0, \varphi) \in [\alpha, \infty) \times C_\tau^m(D)$, then the trivial solution of* (1.1) *is uniformly stable.*

In the work, we need the following theorem, taken from [12, Theorem C, p. 366].

Theorem 1.3.16. *Let w_1 be a continuous non-decreasing function on $[0, M)$ which is zero at 0 and positive on $(0, M)$. Let $\|G(t,\varphi)\| \le B$ for some constant $B > 0$ for all $(t,\varphi) \in [\alpha,\infty) \times C_\tau^m(D)$. If there exists a Lyapunov-Krasovskii functional V such that, whenever $(t_0,\varphi) \in [\alpha,\infty) \times C_\tau^m(D)$ and $x = x(\cdot; t_0, \varphi)$ on $[t_0 - \tau, \infty)$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} V(t, x_t) \le -w_1(\|x(t)\|)$$

for $t \in [t_0, \infty)$, then the trivial solution of (1.1) *is uniformly asymptotically stable.*

2 OPTIMIZATION IN NON-DELAYED CASE

A stabilization problem for a system of differential equations without delay is investigated in this part. Below in parts 2.1 and 2.2 we denote by H a positive number. Parts 2.1 and 2.2 are modifications of parts of [33]. We will use the original concepts and definitions of [33].

2.1 FORMULATION OF THE PROBLEM

Consider a system of non-delayed functional differential equations

$$x'(t) = F(t, x), \tag{2.1}$$

where $F \colon \mathfrak{D}_1 \to \mathbb{R}^m$,

$$\mathfrak{D}_1 := \{(t,x) \in [t_0,\infty) \times \mathbb{R}^m,\ \|x\| \le H\}.$$

Assume that F is continuous and satisfies a local Lipschitz condition with respect to x.
For controllability problems we will consider systems (2.1) with explicitly indicated control functions in the form

$$x'(t) = f(t, x, u), \tag{2.2}$$

where $f \colon \mathfrak{D} \to \mathbb{R}^m$, $f(t, \Theta_m, \Theta_r) = \Theta_m$,

$$\mathfrak{D} := \{(t,x,u) \in [t_0,\infty) \times \mathbb{R}^m \times \mathbb{R}^r,\ \|x\| \le H\}.$$

Applied stabilization problems with the requirement of asymptotic stability of a given motion described by the system of differential equations (2.2) require the best possible quality of the transition process. The best quality criterion is very often formulated minimizing the integral

$$I = \int_{t_0}^{\infty} \omega(t,x,u)\mathrm{d}t, \tag{2.3}$$

where $\omega \colon \mathfrak{D} \to \mathbb{R}$. Frequently, the integrand is assumed to have a quadratic form

$$\omega(t,x,u) = x^T C x + u^T D u$$

with a positive-definite constant $m \times m$ matrix C and an $r \times r$ matrix D.

Problem 2.1.1. The optimal control problem is formulated as follows. Find a function $u = u_0$ such that the quality criterion (2.3) is fufilled and the trivial solution of (2.2) is asymptotically stable.

In other words, let a quality criterion of a process $x(t)$ in the form (2.3) be fixed. It is necessary to find a control function $u = u_0$ ensuring the asymptotic stability of non-perturbed motion $x(t) \equiv 0$ such that, for any other admissible control function $u = u^*$, the inequality

$$\int_{t_0}^{\infty} \omega(t, x, u_0)\mathrm{d}t \leq \int_{t_0}^{\infty} \omega(t, x, u^*)\mathrm{d}t$$

holds. The function $u = u_0$ is called an optimal control function.

Definition 2.1.2. Let $V\colon [t_0, \infty) \times \mathbb{R}^m \to [t_0, \infty)$ be a continuous function. Then, V is called a Lyapunov function if it is a locally positive-definite function, i.e.

$$V(t_0, 0) = 0, \;\; V(t_0, x) > 0 \;\text{ for } \forall (t, x) \in [t_0, \infty) \times U \backslash \{0\}$$

with U being a neighbourhood region around $x = 0$.

Definition 2.1.3. Let V be a Lyapunov function by Definition 2.1.2. V is said to have an infinitesimal upper bound if there exists a continuous non-decreasing function $W\colon [0, H) \longrightarrow \mathbb{R}$, $W(0) = 0$, $W(s) > 0$ if $s \in (0, H)$ such that

$$V(t, x) \leq W(\|x\|)$$

on $[t_0, \infty) \times \mathbb{R}^m$.

Theorem 2.1.4. *If a function V can be found for the differential equations of the disturbed motion (2.2) satisfying Definition 2.1.2 for which the derivative with respect to time based on these equations $\mathrm{d}V/\mathrm{d}t$ is negative and the function V itself permits an infinitesimal upper bound, then the undisturbed motion is asymptotically stable.*

2.2 MALKIN'S RESULT

Define an auxiliary function $B\colon \mathfrak{D}_2 \to \mathbb{R}$,

$$\mathfrak{D}_2 \colon = \{(v, t, x, u) \in \mathbb{R} \times [t_0, \infty) \times \mathbb{R}^m \times \mathbb{R}^r, \; \|x\| \leq H\},$$

by the formula

$$B\left(V, t, x, u\right) := \frac{\mathrm{d}V(t, x)}{\mathrm{d}t} + \omega(t, x, u),$$

where V is a Lyapunov function.
Let us formulate the main theorem of optimal stabilization presented in [33, p. 475–514] utilizing the second Lyapunov method as applied to ordinary differential equations.

Theorem 2.2.1. *Assume that, for the system of differential equations (2.2), there exists a Lyapunov function $V_0(t, x)$ having an infinitesimal upper bound and a function u_0 such that*
i) the function $\omega(t, x, u)$ is positive-definite for every $t \geq t_0$, $\|x\| < H$, $u \in \mathbb{R}^r$;

ii) $B\left(V_0, t, x, u_0\right) \equiv 0$;

iii) $B\left(V_0, t, x, u\right) \geq 0$ for any $u \not\equiv u_0$.

Then, the function u_0 is a solution of the optimal control problem and

$$\int_{t_0}^{\infty} \omega(t, x, u_0)\mathrm{d}t = \min_u \left[\int_{t_0}^{\infty} \omega(t, x, u)\mathrm{d}t\right] = V_0(t_0, x).$$

2.3 APPLICATIONS TO LINEAR EQUATIONS AND SYSTEMS

In this part, we apply Theorem 2.2.1 to a class of ordinary differential equations and their systems. The results derived are not included in [33].

I. Consider a scalar equation

$$x'(t) = ax(t) + bu, \tag{2.4}$$

where a and b ($b \neq 0$) are real constants. Together with the equation (2.4), we will consider the quality criterion (2.3) with

$$\omega(t, x, u) = cx^2(t) + du^2,$$

where $c > 0$, $d > 0$, that is,

$$I = \int_{t_0}^{\infty} (cx^2(t) + du^2)\mathrm{d}t. \tag{2.5}$$

Theorem 2.3.1. *If, for the optimal control problem* (2.4), (2.5), *a Lyapunov function in the form*

$$V(t, x) = hx^2(t),$$

where

$$h = \frac{ad + \sqrt{a^2d^2 + b^2cd}}{b^2}$$

is used, then the optimal control function is

$$u_0 = -\frac{hb}{d}x(t).$$

II. Consider a linear system with a scalar control function:

$$x'(t) = Ax(t) + bu, \tag{2.6}$$

where $A \in \mathbb{R}^{m\times m}$, $b \in \mathbb{R}^m$, $x(t) \in \mathbb{R}^m$, $u \in \mathbb{R}$. We need to find a control function $u = u_0$ for which the system (2.6) is asymptotically stable and a given integral quality criterion

$$I = \int_{t_0}^{\infty} \left(x^T(t)Cx(t) + du^2\right)\mathrm{d}t \tag{2.7}$$

has a minimum value provided that C is an $m \times m$ symmetric positive-definite matrix and $d > 0$. We will use a Lyapunov function

$$V(t, x) = x^T(t)Hx(t),$$

where H is an $m \times m$ positive-definite symmetric matrix.
In the sequel, define $\Theta_{\kappa\times\kappa}$ as a zero $\kappa \times \kappa$ matrix.

Theorem 2.3.2. *Assume that there exists a positive-definite symmetric matrix H satisfying the matrix equation*

$$A^TH + HA + C - \frac{1}{d}Hbb^TH = \Theta_{m\times m}.$$

Then, the optimal stabilization control function $u = u_0$ of the problem (2.6), (2.7) *exists and*

$$u_0 = -\frac{1}{d}b^THx(t).$$

III. As the next application consider a system:

$$x'(t) = Ax(t) + Pu, \tag{2.8}$$

where $A \in \mathbb{R}^{m\times m}$, $P \in \mathbb{R}^{m\times r}$, $x(t) \in \mathbb{R}^m$, $u \in \mathbb{R}^r$. We need to find an optimal control function $u = u_0$ for which the system is asymptotically stable and an integral quality criterion

$$I = \int_{t_0}^{\infty} \left(x^T(t)Cx(t) + u^T Du \right) \mathrm{d}t \tag{2.9}$$

takes a minimum value provided that $C \in \mathbb{R}^{m\times m}$ is a symmetric, positive-definite matrix and D is a diagonal control matrix, $D = \mathrm{diag}\{d_j\}$, $d_j > 0$, $j = 1, \ldots, r$.
We will use a Lyapunov function

$$V(t, x) = x^T(t)Hx(t),$$

where H is an $m \times m$ positive-definite symmetric matrix.

Theorem 2.3.3. *Assume that there exists a positive-definite symmetric matrix H satisfying the matrix equation*

$$A^T H + HA + C - HPD^{-1}P^T H = \Theta_{m\times m}.$$

Then, the optimal stabilization control function $u = u_0$ of the problem (2.8), (2.9) *exists and*

$$u_0 = -D^{-1}P^T Hx(t).$$

3 OPTIMIZATION IN DELAYED CASE

In this part, we will consider systems of delayed scalar equations with constant coefficients. For such equations, we will find control functions theoretically and, in specific cases, by using the formulas obtained. The results of this chapter are new.

3.1 FORMULATION OF THE PROBLEM

Consider an arbitrary dynamic process and assume that it can be described by a system of functional differential equations of delayed type

$$x'(t) = F(t, x_t), \tag{3.1}$$

where $F \colon \mathcal{D}_1 \to \mathbb{R}^m$,

$$\mathcal{D}_1 := \{(t, x_t) \in [t_0, \infty) \times C_\tau^m,\ \|x_t\|_\tau \le M_x\}$$

and M_x is a given positive constant. Let the functional F be continuous, locally Lipschitzian and quasi-bounded. Together with (3.1), we consider the initial problem

$$x_{t_*} = \varphi, \tag{3.2}$$

where $t_* \ge t_0$, and $\varphi \in C_\tau^m$.
Our goal is to be able to control the process. Consider a process $x \colon [t_0 - \tau, \infty) \to \mathbb{R}^m$, controlled by means of a control function (or control functional) $u = u(t, x_t)$, where

$$u \colon \mathcal{D}_1 \to \mathbb{R}^r,\ \ u(t, \theta_m^*) = \theta_r$$

such that $\|u(t, x_t)\| \le M_u$, $(t, x_t) \in \mathcal{D}_1$, M_u is a given positive constant, and assuming that u is continuous, locally Lipschitzian and quasi-bounded. Assume that the process can be modelled by a system of differential equations of delayed type

$$x'(t) = f(t, x_t, u), \quad t \ge t_0, \tag{3.3}$$

where $f\colon \mathcal{D} \to \mathbb{R}^m$,

$$\mathcal{D} := \{(t, x_t, u) \in [t_0, \infty) \times C_\tau^m \times \mathbb{R}^r, \ \|x_t\|_\tau \le M_x, \ \|u\| \le M_u\}$$

and $\|u\|$ is defined as in (1.3). Assume that

$$f(t, \theta_m^*, \theta_r) = \theta_m$$

and that f is continuous, locally Lipschitzian and quasi-bounded. Let, moreover, for a constant $K_1 \ge 0$, $\|f(t, x_t, u)\| \le K_1$ whenever $(t, x_t, u) \in \mathcal{D}$.
If we specify $F(t, x_t) := f(t, x_t, u)$, where $u = u(t, x_t)$, then the system

$$x'(t) = f(t, x_t, u(t, x_t)), \quad t \ge t_0 \tag{3.4}$$

is a particular case of the system (3.1) and (1.1) and, consequently, the auxiliary concepts formulated for (1.1) in part 1.3 can be applied to the system (3.4) as well.
In what follows, we will assume, without loss of generality, that the constant M_x is so large that the below solutions of the system (3.4), defined on $[t_0 - \tau, \infty)$, satisfy $\|x(t)\| \le M_x$, $t \in [t_0 - \tau, \infty)$.
The problem under consideration is formulated as follows.

Problem 3.1.1. Find a control function $u = u_0(t, x_t)$ such that the zero solution $x(t) = \theta_m$, $t \ge t_0 - \tau$ of the system

$$x'(t) = f(t, x_t, u_0(t, x_t)), \quad t \ge t_0, \tag{3.5}$$

is asymptotically stable and, for an arbitrary solution $x\colon [t_0 - \tau, \infty) \to \mathbb{R}^m$ of (3.5), satisfying $\|x_{t_0}\|_\tau \le \eta$, η is a sufficiently small positive number such that $\eta \le M_x$, the integral quality criterion

$$I = \int_{t_0}^{\infty} \omega\left(t, x_t, u_0(t, x_t)\right) \mathrm{d}t, \tag{3.6}$$

where $\omega\colon \mathcal{D} \to \mathbb{R}$ is a given positive-definite functional, exists and attains the minimum value. This means that, for an arbitrary control function $u = u^*(t, x_t)$ such that the zero solution $x(t) = \theta_m$, $t \ge t_0 - \tau$ of system

$$x'(t) = f(t, x_t, u^*(t, x_t)), \quad t \ge t_0, \tag{3.7}$$

is asymptotically stable, we have

$$\int_{t_0}^{\infty} \omega\left(t, x_t, u_0(t, x_t)\right) \mathrm{d}t \le \int_{t_0}^{\infty} \omega\left(t, x_t^*, u^*(t, x_t^*)\right) \mathrm{d}t,$$

where $x\colon [t_0 - \tau, \infty) \to \mathbb{R}^m$ is the solution of (3.5) defined by the initial problem (3.2) where $t_* := t_0$, and $x^*\colon [t_0 - \tau, \infty) \to \mathbb{R}^m$ is the solution of (3.7) defined by the same initial problem. The initial function φ in (3.2) is arbitrary except for the assumption $\|\varphi\|_\tau \le \eta$.

Remark 3.1.2. Modifying the above Definition 1.3.11 of a positive-definite functional to the functional $\omega\colon \mathcal{D} \to \mathbb{R}$ used in (3.6), we specify that ω is a positive-definite functional if there exists a continuous non-decreasing function $w^*(y_1, y_2)$ defined on the set $\mathcal{S} := \{[0,\infty)\times[0,\infty)\}$ such that $w^*(0,0) = 0$ and $w^*(y_1, y_2) > 0$ if $(y_1, y_2) \in \mathcal{S} \setminus \{(0,0)\}$, and

$$\omega(t, x_t, u) \geq w^*(\|x(t)\|, \|u\|), \;\; t \geq t_0$$

whenever $(t, x_t, u) \in \mathcal{D}$. The non-decreasing property of w^* means that

$$w^*(y_1, y_2) \leq w^*(\bar{y}_1, \bar{y}_2)$$

whenever $y_1 \leq \bar{y}_1$, $y_2 \leq \bar{y}_2$ and $(y_1, y_2) \in \mathcal{S}$, $(\bar{y}_1, \bar{y}_2) \in \mathcal{S}$.

Remark 3.1.3. We call the function $u_0(t, x_t)$ solving Problem 3.1.1 the *optimal stabilization control function.* Moreover, the problem of minimizing the integral I by an optimal stabilization control function, as described in Problem 3.1.1, can be formulated more succinctly using the following notation

$$I = \min_u \int_{t_0}^{\infty} \omega(t, x_t, u(t, x_t))\mathrm{d}t.$$

Problem 3.1.1 extends to delayed differential equations Problem II formulated for ordinary differential equations in Malkin's book [33, p. 479]. This problem is formulated above in part 2.1 as well (Problem 2.1.1).

Remark 3.1.4. The optimal stabilization control function $u_0(t, x_t)$, solving Problem 3.1.1, as well as every other control function $u(t, x_t)$ mentioned in the work, is actually a function of the variable t. Therefore, without loss of generality, we sometimes use $u_0(t)$, $u(t)$ or u_0, u for short if there is no danger of ambiguity.

3.2 GENERALIZATION OF MALKIN'S RESULT

To solve the problem we are motivated by Malkin's approach, presented in Section 2.2.
Define a functional $B\colon \mathcal{D}_2 \to \mathbb{R}$,

$$\mathcal{D}_2 := \{(v, t, x_t, u) \in \mathbb{R} \times [t_0, \infty) \times C_\tau^m \times \mathbb{R}^r,\ \|x_t\|_\tau \leq M_x,\ \|u\| \leq M_u\},$$

by the formula

$$B\,(V, t, x_t, u) := \frac{\mathrm{d}V(t, x_t)}{\mathrm{d}t} + \omega(t, x_t, u),$$

where V is defined by (1.4) and the derivative of V is computed as in Definition 1.3.14 provided that x is an arbitrary fixed solution of the system (3.3).
The next theorem is a generalization of Theorem 2.2.1 for the case of delayed differential equations.

Theorem 3.2.1. *Assume that, for the system of differential equations of delayed type* (3.3), *there exists a Lyapunov-Krasovskii functional $V(t, x_t)$ and a control function $u_0(t, x_t)$ such that*

i) the functional $\omega\colon \mathcal{D} \to \mathbb{R}$ is positive-definite;

ii) the identity

$$B(V, t, x_t, u_0(t, x_t)) \equiv 0$$

holds on $[t_0, \infty)$ *for every solution* $x\colon [t_0-\tau, \infty) \to \mathbb{R}^m$ *of the system* (3.3)*, where* $u = u_0(t, x_t)$*;*

iii) the inequality $B(V, t, x_t, u) \geq 0$ *holds on* $[t_0, \infty)$ *for every solution* $x\colon [t_0 - \tau, \infty) \to \mathbb{R}^m$ *of the system* (3.3) *with arbitrary fixed control function* $u = u(t, x_t)$*.*

Then, the function $u_0(t, x_t)$ *is the optimal stabilization control function solving Problem 3.1.1, that is,*

$$I = \min_u \int_{t_0}^{\infty} \omega(t, x_t, u(t, x_t))\mathrm{d}t = \int_{t_0}^{\infty} \omega(t, x_t, u_0(t, x_t))\mathrm{d}t$$

and, moreover,

$$\int_{t_0}^{\infty} \omega(t, x_t, u_0(t, x_t))\mathrm{d}t = V(t_0, x_{t_0}).$$

Remark 3.2.2. Theorem 3.2.1 is an extension to delayed differential equations of Theorem IV in Malkin's book [33, p. 485] formulated there for ordinary differential equations. Optimal problems for delayed differential equations with integral quality criteria are often considered for a finite upper limit in an integral quality criterion I and are, in general, not applicable to the case of this limit being infinite (we refer, for example, to [6,8,9,18,19,24,25,32,38,39,43,48] and to the references therein). In [16, 23], the quality criteria are considered in an integral form with an infinite upper limit. Unlike our investigation, a control law is searching in the prescribed class of functionals. In [10], an integral quality criterion with an infinite upper limit is used for solving an optimal control problem, but a weight function of an exponential type is used to preserve its convergence.

3.3 APPLICATION TO LINEAR EQUATIONS AND SYSTEMS

I. Consider linear scalar equations with constant coefficients and a single delay

$$x'(t) = ax(t) + bx(t - \tau) + cu, \tag{3.8}$$

where a, $b \neq 0$, c are real constants, $\tau > 0$ is a delay and u is a control function. Together with the equation (3.8), we will consider a quality criterion (3.6) with

$$\omega(t, x_t, u) = \alpha x^2(t) + 2\beta x(t)x(t - \tau) + \gamma x^2(t - \tau) + \delta u^2,$$

i.e., (3.6) being a quadratic criterion

$$I = \int_{t_0}^{\infty} \Big(\alpha x^2(t) + 2\beta x(t)x(t - \tau) + \gamma x^2(t - \tau) + \delta u^2\Big)\mathrm{d}t, \tag{3.9}$$

with

$$\alpha > 0,\ \alpha\gamma - \beta^2 > 0,\ \delta > 0.$$

Theorem 3.3.1. *If, for the optimal control problem* (3.8)*,* (3.9)*, a Lyapunov-Krasovskii functional is used in the form*

$$V(t, x_t) = hx^2(t) + d\int_{t-\tau}^{t} x^2(s)\mathrm{d}s,\ h > 0,\ d > 0,$$

with $h = -\beta/b$ *(*$\beta b < 0$*),* $d = \gamma$*,*

$$\delta(2ha + d + \alpha) - h^2c^2 = 0,$$

then the optimal stabilization control function u_0 *equals*

$$u_0 = -\frac{hc}{\delta}x(t).$$

II. Consider linear scalar equations with constant coefficients and delays

$$x'(t) = ax(t) + \sum_{i=1}^{n} b_i x(t-\tau_i) + cu, \ \ t \geq 0, \tag{3.10}$$

where a, b_i and c are real constants, $i = 1, \ldots, n$, $\tau_1 < \tau_2 < \cdots < \tau_n = \tau$ are delays and u is a control function.
Together with the equation (3.10), we will consider a quality criterion (3.6) with

$$\omega(t, x_t, u) = \sum_{i=0}^{n} \alpha_i x^2(t-\tau_i) + 2\sum_{i=1}^{n} \beta_i x(t)x(t-\tau_i) + \gamma u^2,$$

where $\tau_0 = 0$, i.e., (3.6) being a quadratic criterion

$$I = \int_{t_0}^{\infty} \left(\sum_{i=0}^{n} \alpha_i x^2(t-\tau_i) + 2\sum_{i=1}^{n} \beta_i x(t)x(t-\tau_i) + \gamma u^2 \right) \mathrm{d}t, \tag{3.11}$$

where α_0, α_i, β_i $(i = 1, \ldots, n)$ and $\gamma > 0$ are constants and the matrix

$$\begin{pmatrix} \alpha_0 & \beta_1 & \beta_2 & \ldots & \beta_n \\ \beta_1 & \alpha_1 & 0 & \ldots & 0 \\ \beta_2 & 0 & \alpha_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_n & 0 & 0 & \ldots & \alpha_n \end{pmatrix}$$

is positive-definite.

Theorem 3.3.2. *Let*

$$-\beta_i/b_i = h > 0, \ \ i = 1, \ldots, n.$$

If for the optimal control problem (3.10), (3.11) *a Lyapunov-Krasovskii functional is used in the form*

$$V(t, x_t) = hx^2(t) + \sum_{i=1}^{n} d_i \int_{t-\tau_i}^{t} x^2(s)\mathrm{d}s, \ \ h > 0, \ d_i > 0,$$

with

$$d_i = \alpha_i$$

and if

$$\gamma \left(2ha + \sum_{i=1}^{n} d_i + \alpha_0 \right) - h^2c^2 = 0,$$

then the optimal stabilization control function u_0 equals

$$u_0 = -\frac{hc}{\gamma} x(t).$$

III. Consider linear systems with constant coefficients and a single constant delay

$$x'(t) = A_0 x(t) + A_1 x(t-\tau) + bu, \tag{3.12}$$

where A_0, A_1 are $m \times m$ constant matrices, $b \in \mathbb{R}^m$, $u \in \mathbb{R}$, and a quality criterion (3.6) with

$$\begin{aligned}\omega(t, x_t, u) = {} & x^T(t)C_{11}x(t) + x^T(t)C_{12}x(t-\tau) \\ & + x^T(t-\tau)C_{21}x(t) + x^T(t-\tau)C_{22}x(t-\tau) + du^2,\end{aligned}$$

where $m \times m$ matrices C_{11}, C_{22} and an $2m \times 2m$ matrix

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \tag{3.13}$$

are positive-definite and symmetric, $C_{21} = C_{12}^T$ and $d > 0$, i.e., (3.6) is a quadratic criterion

$$I = \int_{t_0}^{\infty} \Big(x^T(t)C_{11}x(t) + x^T(t)C_{12}x(t-\tau) \\ + x^T(t-\tau)C_{21}x(t) + x^T(t-\tau)C_{22}x(t-\tau) + du^2 \Big)\,\mathrm{d}t. \tag{3.14}$$

We will employ a Lyapunov-Krasovskii functional

$$V(t, x_t) = x^T(t)Hx(t) + \int_{t-\tau}^{t} x^T(s)Gx(s)\mathrm{d}s, \tag{3.15}$$

where H and G are $m \times m$, constant, positive-definite and symmetric matrices.

Theorem 3.3.3. *Assume that there exists a positive-definite symmetric $m \times m$ matrix H satisfying a matrix equation*

$$A_0^T H + HA_0 + C_{11} + C_{22} - \frac{1}{d}Hbb^T H = \Theta_{m\times m}. \tag{3.16}$$

If, moreover,

$$HA_1 + C_{12} = \Theta_{m\times m},$$

the optimal stabilization control function $u = u_0$ of the problem (3.12), (3.14) *exists and*

$$u_0 = -\frac{1}{d}b^T Hx(t).$$

IV. Consider linear systems with constant coefficients and a single constant delay

$$x'(t) = A_0x(t) + A_1x(t-\tau) + Pu, \tag{3.17}$$

where A_0, A_1 are $m \times m$ constant matrices, $P \in \mathbb{R}^{m\times r}$, $u \in \mathbb{R}^r$, and a quality criterion (3.6)

$$I = \int_{t_0}^{\infty} (x^T(t)C_{11}x(t) + x^T(t)C_{12}x(t-\tau) \\ + x^T(t-\tau)C_{21}x(t) + x^T(t-\tau)C_{22}x(t-\tau) + u^T Du)\mathrm{d}t, \tag{3.18}$$

where $m \times m$ matrices C_{11}, C_{22} and an $2m \times 2m$ matrix (3.13), i.e.,

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

are positive-definite and symmetric, $C_{21} = C_{12}^T$ and D is a diagonal matrix, $D = \mathrm{diag}\{d_j\}$, $d_j > 0$, $j = 1, \dots, r$.
We will use a Lyapunov-Krasovskii functional (3.15), that is

$$V(t, x_t) = x^T(t)Hx(t) + \int_{t-\tau}^{t} x^T(s)Gx(s)\mathrm{d}s,$$

where H and G are $m \times m$ constant, positive-definite and symmetric matrices.

Theorem 3.3.4. *Assume that there exists a positive-definite symmetric matrix H satisfying the matrix equation*

$$A_0^T H + HA_0 + C_{11} + C_{22} - HPD^{-1}P^T H = \Theta_{m\times m}. \tag{3.19}$$

If, moreover,

$$HA_1 + C_{12} = \Theta_{m\times m},$$

the optimal stabilization control function $u = u_0$ of the problem (3.17), (3.18) *exists and*

$$u_0 = -D^{-1}P^T Hx(t).$$

V. In this part, we consider systems of linear differential equations with delays

$$x'(t) = \sum_{i=0}^{n} A_i x(t-\tau_i) + cu, \quad t \geq t_0, \tag{3.20}$$

where A_i, $i = 0, \dots, n$ are $m\times m$ real matrices, $c \in \mathbb{R}^m$, $0 = \tau_0 < \tau_1 < \dots < \tau_n$, $x\colon [t_0-\tau, \infty) \to \mathbb{R}^m$, $t_0 \in \mathbb{R}$ and $u \in \mathbb{R}$ is a control function. Set $\tau := \tau_n$. A minimization problem (3.6) with

$$\omega(t, x_t, u) := \sum_{i=0}^{n} x^T(t-\tau_i)C_{ii}x(t-\tau_i) + \sum_{i=1}^{n} x^T(t)C_{0i}x(t-\tau_i) + \sum_{i=1}^{n} x^T(t-\tau_i)C_{i0}x(t) + du^2 \tag{3.21}$$

will be solved for the system (3.20), where constant symmetric $m \times m$ matrices C_{ii} and an auxiliary matrix

$$C = \begin{pmatrix} C_{00} & C_{01} & \dots & C_{0n} \\ C_{10} & C_{11} & \dots & C_{1n} \\ \vdots & \vdots & \ddots & \\ C_{n0} & C_{n1} & \dots & C_{nn} \end{pmatrix},$$

(with $C_{ij} = C_{ji} = \Theta_{m\times m}$, $i > j \geq 1$, $i, j = 1, \dots, n$) are positive-definite, C_{0i} and C_{i0}, $C_{0i} = C_{i0}^T$ are $m \times m$ constant matrices, $d > 0$. We will employ a Lyapunov-Krasovskii functional

$$V(t, x_t) = x^T(t)Hx(t) + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} x^T(s)G_i x(s)\mathrm{d}s, \tag{3.22}$$

where $m \times m$ matrices H and G_i, $i = 1, \dots, n$ are constant, positive-definite and symmetric.

Theorem 3.3.5. *Assume that the matrix C is positive-definite and there exists a positive-definite symmetric matrix H satisfying the matrix equation*

$$A_0^T H + HA_0 + \sum_{i=0}^{n} C_{ii} - \frac{1}{d}Hcc^T H = \Theta_{m\times m}. \tag{3.23}$$

If, moreover,

$$A_i^T H + C_{i0} = \Theta_{m\times m}, \ i = 1, \dots, n$$

then the optimal stabilization control function of the problem (3.20), (3.21) *exists and equals*

$$u_0 = -\frac{1}{d}c^T Hx(t).$$

VI. In this part, we consider systems of linear differential equations with delays

$$x'(t) = \sum_{i=0}^{n} A_i x(t-\tau_i) + Cu, \ \ t \geq t_0, \tag{3.24}$$

where A_i, $i = 0, \dots, n$ are $m \times m$ real matrices, C is an $m \times r$ real matrix, $0 = \tau_0 < \tau_1 < \dots < \tau_n$, $x\colon [t_0 - \tau, \infty) \to \mathbb{R}^m$, $t_0 \in \mathbb{R}$ and $u\colon \mathcal{D}_1 \to \mathbb{R}^r$ is a control function. Set $\tau := \tau_n$. A minimization problem

$$I = \min_u \int_{t_0}^{\infty} \omega\left(t, x_t, u\right) \mathrm{d}t, \tag{3.25}$$

where

$$\begin{aligned}\omega\left(t, x_t, u\right) := \sum_{i=0}^{n} x^T(t-\tau_i) C_{ii} x(t-\tau_i) + \sum_{i=1}^{n} x^T(t) C_{0i} x(t-\tau_i) \\ + \sum_{i=1}^{n} x^T(t-\tau_i) C_{i0} x(t) + \sum_{i=0}^{n} u^T D_i x(t-\tau_i) \\ + \sum_{i=0}^{n} x^T(t-\tau_i) D_i^* u + u^T D u\end{aligned} \tag{3.26}$$

will be solved for the system (3.24), where C_{ii} are $m \times m$ constant symmetric matrices, C_{0i} and C_{i0}, $C_{0i} = C_{i0}^T$ are $m \times m$ constant matrices, D is an $r \times r$ symmetric matrix and D_i, D_i^*, $D_i = (D_i^*)^T$ are $r \times m$ and $m \times r$ constant matrices, respectively. Define auxiliary matrices $C_{ij} = C_{ji} = \Theta_{m \times m}$, $(i > j \geq 1,\ i, j = 1, \dots, n)$. Let $X(t)$ be an $[(n+1)m + r] \times 1$ vector defined by the formula

$$X(t) = (x^T(t), x^T(t-\tau_1), \dots, x^T(t-\tau_n), u)^T$$

and

$$\mathcal{C} = \begin{pmatrix} C_{00} & C_{01} & \dots & C_{0n} & D_0^* \\ C_{10} & C_{11} & \dots & C_{1n} & D_1^* \\ \vdots & \vdots & \ddots & \vdots & \\ C_{n0} & C_{n1} & \dots & C_{nn} & D_n^* \\ D_0 & D_1 & \dots & D_n & D \end{pmatrix}.$$

Then, the formula (3.26) can be written in the form

$$\omega\left(t, x_t, u\right) = X^T(t) \mathcal{C} X(t).$$

Below we assume that the matrix $\mathcal{C}$ is positive-definite, that is, the functional $\omega(t, x_t, u)$ is positive-definite. In the following, we will employ a Lyapunov-Krasovskii functional (3.22), that is

$$V(t, x_t) = x^T(t) H x(t) + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} x^T(s) G_i x(s) \mathrm{d}s,$$

where $m \times m$ matrices H and G_i, $i = 1, \dots, n$ are constant, positive-definite and symmetric. Their elements will be defined in the formulation of the theorem below.

Theorem 3.3.6. *Assume that the matrix $\mathcal{C}$ is positive-definite and there exist positive-definite symmetric matrices H and G_i, $i = 1, \ldots, n$, satisfying*

$$A_0^T H + HA_0 + C_{00} + \sum_{i=1}^{n} G_i - [HC + D_0^*]\, D^{-1}\Big[C^T H + D_0\Big] = \Theta_{m\times m}, \tag{3.27}$$

$$A_i^T H + C_{i0} - D_i^* D^{-1}\Big[C^T H + D_0\Big] = \Theta_{m\times m}, \quad i = 1, \ldots, n,$$

$$G_i - C_{ii} - D_i^* D^{-1} D_i = \Theta_{m\times m}, \quad i = 1, \ldots, n.$$

If, moreover,

$$D_i^* D^{-1} D_j = \Theta_{m\times m},\ i, j = 1, \ldots, n,\ i \neq j,$$

then the optimal stabilization control function of the problem (3.24)–(3.26) *exists and equals*

$$u_0 = -D^{-1}\Big[C^T H + D_0\Big] x(t) - D^{-1} \sum_{i=1}^{n} D_i x(t - \tau_i).$$

As a particular case of Theorem 3.3.6, consider the system (3.24) with the quality criterion (3.25) where matrices D_i, D_i^*, $i = 0, \ldots, n$ are zero matrices, that is, let

$$\omega\,(t, x_t, u) := \sum_{i=0}^{n} x^T(t - \tau_i) C_{ii} x(t - \tau_i) + \sum_{i=1}^{n} x^T(t) C_{0i} x(t - \tau_i)$$
$$+ \sum_{i=1}^{n} x^T(t - \tau_i) C_{i0} x(t) + u^T D u. \tag{3.28}$$

Then, the following holds.

Theorem 3.3.7. *Assume that the matrix $\mathcal{C}$ is positive-definite and there exist positive-definite symmetric matrices H and G_i, $i = 1, \ldots, n$, satisfying*

$$A_0^T H + HA_0 + C_{00} + \sum_{i=1}^{n} G_i - HCD^{-1}C^T H = \Theta_{m\times m}, \tag{3.29}$$

$$A_i^T H + C_{i0} = \Theta_{m\times m}, \quad i = 1, \ldots, n.$$

If, moreover

$$G_i = C_{ii}, \quad i = 1, \ldots, n,$$

then the optimal stabilization control function of the problem (3.24), (3.25), (3.28) *exists and equals*

$$u_0 = -D^{-1} C^T H x(t).$$

4 CONCLUSIONS

The thesis considers the problem of optimal stabilization for ordinary and functional differential systems. It is based on the result [Theorem 2.2.1, page 11] given in Malkin's book [33, Theorem IV, page 485]. The book [33] is a revised edition of the book [34] and, furthermore, contains new parts - Additions I–IV, prepared by Malkin's followers led by academician N. Krasovskii. In the thesis, first Theorem 2.2.1 was applied to some classes of linear non-delayed differential equations and then the previous result was extended to delayed differential equations and systems. If the delay vanishes ($\tau = 0$), our results reduce back to those already known from [33].

The main result of the thesis is Theorem 3.2.1 (page 15), which solves the problem of minimizing an integral quality criterion. In order to solve this problem, we find an optimal stabilization control function, which simultaneously guarantees the asymptotic stability of a given system of differential equations. The result obtained is successfully applied to certain classes of linear differential equations with delays.

The problems and derived results, formulated in the thesis, can serve as a motivation for further research. For example, in the thesis, the assumption *iii*) ($B(V, t, x_t, u_0) = 0$) from Theorem 3.2.1 (page 15) is considered only in cases explicitly solvable with respect to u_0. It is also an open question if the theory of implicit functions can be applied to more complicated cases and, consequently, if the results obtained in the thesis can be extended. Another challenge is to apply the results to linear systems with variable coefficients, first in the case of the coefficients being almost constant (for $t \to \infty$).

As a topic for future research, investigation of the solvability of the matrix equations (in the thesis, for example, equations (3.16), (3.19), (3.23), (3.27)) with respect to the matrix H can be suggested as well.

Application of the main result to linear systems leads to complicated systems of nonlinear equations, which determine the elements of the matrix H that has a crucial role in the formulated criteria. In the examples of this thesis, we sometimes overcome this circumstance by using a suitable software. That is why it could be useful to create a special program for solving certain classes of the problems considered.

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ABSTRACT

The present thesis deals with processes controlled by systems of delayed differential equations

$$x'(t) = f(t, x_t, u), \quad t \geq t_0$$

where $t_0 \in \mathbb{R}$, f is defined on a subspace of $[t_0, \infty) \times C_\tau^m \times \mathbb{R}^r$, $m, r \in \mathbb{N}$, $C_\tau^m = C([-\tau, 0], \mathbb{R}^m)$, $\tau > 0$, $x_t(\theta) := x(t+\theta)$, $\theta \in [-\tau, 0]$, $x\colon [t_0 - \tau, \infty) \to \mathbb{R}^m$. Under the assumption $f(t, \theta_m^*, \theta_r) = \theta_m$, where $\theta_m^* \in C_\tau^m$ is a zero vector-function, θ_r and θ_m are r and m-dimensional zero vectors, a control function $u = u(t, x_t)$, $u\colon [t_0, \infty) \times C_\tau^m \to \mathbb{R}^r$, $u(t, \theta_m^*) = \theta_r$ is determined such that the zero solution $x(t) = \theta_m$, $t \geq t_0 - \tau$ of the system is asymptotically stable and, for an arbitrary solution $x = x(t)$, the integral

$$\int_{t_0}^{\infty} \omega\left(t, x_t, u(t, x_t)\right) \mathrm{d}t,$$

where ω is a positive-definite functional, exists and attains its minimum value in a given sense. To solve this problem, Malkin's approach to ordinary differential systems is extended to delayed functional differential equations and Lyapunov's second method is applied. The results are illustrated by examples and applied to some classes of delayed linear differential equations.

ABSTRAKT

Dizertační práce se zabývá procesy, které jsou řízeny systémy zpožděných diferenciálních rovnic

$$x'(t) = f(t, x_t, u), \quad t \geq t_0$$

kde $t_0 \in \mathbb{R}$, funkce f je definována v jistém podprostoru množiny $[t_0, \infty) \times C_\tau^m \times \mathbb{R}^r$, $m, r \in \mathbb{N}$, $C_\tau^m = C([-\tau, 0], \mathbb{R}^m)$, $\tau > 0$, $x_t(\theta) := x(t + \theta)$, $\theta \in [-\tau, 0]$, $x\colon [t_0 - \tau, \infty) \to \mathbb{R}^m$. Za předpokladu $f(t, \theta_m^*, \theta_r) = \theta_m$, kde $\theta_m^* \in C_\tau^m$ je nulová vektorová funkce, θ_r a θ_m jsou r a m-dimenzionální nulové vektory, je řídící funkce $u = u(t, x_t)$, $u\colon [t_0, \infty) \times C_\tau^m \to \mathbb{R}^r$, $u(t, \theta_m^*) = \theta_r$ určena tak, že nulové řešení $x(t) = \theta_m$, $t \geq t_0 - \tau$ systému je asymptoticky stabilní a pro libovolné řešení $x = x(t)$ integrál

$$\int_{t_0}^{\infty} \omega\left(t, x_t, u(t, x_t)\right) \mathrm{d}t,$$

kde ω je pozitivně definitní funkcionál, existuje a nabývá své minimální hodnoty v daném smyslu. Pro řešení tohoto problému byla Malkinova metoda pro obyčejné diferenciální systémy rozšířena na zpožděné funkcionální diferenciální rovnice a byla použita druhá metoda Lyapunova. Výsledky jsou ilustrovány příklady a aplikovány na některé třídy zpožděných lineárních diferenciálních rovnic.