BRNO UNIVERSITY OF TECHNOLOGY

Faculty of Electrical Engineering
and Communication

DOCTORAL THESIS

Brno, 2018

Mgr. Hanna Demchenko

BRNO UNIVERSITY OF TECHNOLOGY
VYSOKÉ UČENÍ TECHNICKÉ V BRNĚ

FACULTY OF ELECTRICAL ENGINEERING AND COMMUNICATION
FAKULTA ELEKTROTECHNIKY
A KOMUNIKAČNÍCH TECHNOLOGIÍ

DEPARTMENT OF MATHEMATICS
ÚSTAV MATEMATIKY

OPTIMIZATION OF DELAYED DIFFERENTIAL SYSTEMS BY LYAPUNOV'S DIRECT METHOD
OPTIMALIZACE DIFERENCIÁLNÍCH SYSTÉMŮ SE ZPOŽDĚNÍM UŽITÍM PŘÍMÉ METODY LYAPUNOVA

DOCTORAL THESIS
DIZERTAČNÍ PRÁCE

AUTHOR
AUTOR PRÁCE

Mgr. Hanna Demchenko

SUPERVISOR
ŠKOLITEL

prof. RNDr. Josef Diblík, DrSc.

BRNO 2018

ABSTRACT

The present thesis deals with processes controlled by systems of delayed differential equations

$$x'(t) = f(t, x_t, u), \quad t \geq t_0$$

where $t_0 \in \mathbb{R}$, f is defined on a subspace of $[t_0, \infty) \times C_\tau^m \times \mathbb{R}^r$, $m, r \in \mathbb{N}$, $C_\tau^m = C([-\tau, 0], \mathbb{R}^m)$, $\tau > 0$, $x_t(\theta) := x(t+\theta)$, $\theta \in [-\tau, 0]$, $x\colon [t_0 - \tau, \infty) \to \mathbb{R}^m$. Under the assumption $f(t, \theta_m^*, \theta_r) = \theta_m$, where $\theta_m^* \in C_\tau^m$ is a zero vector-function, θ_r and θ_m are r and m-dimensional zero vectors, a control function $u = u(t, x_t)$, $u\colon [t_0, \infty) \times C_\tau^m \to \mathbb{R}^r$, $u(t, \theta_m^*) = \theta_r$ is determined such that the zero solution $x(t) = \theta_m$, $t \geq t_0 - \tau$ of the system is asymptotically stable and, for an arbitrary solution $x = x(t)$, the integral

$$\int_{t_0}^{\infty} \omega\left(t, x_t, u(t, x_t)\right) \mathrm{d}t,$$

where ω is a positive-definite functional, exists and attains its minimum value in a given sense. To solve this problem, Malkin's approach to ordinary differential systems is extended to delayed functional differential equations and Lyapunov's second method is applied. The results are illustrated by examples and applied to some classes of delayed linear differential equations.

KEYWORDS

optimal stabilization, control function, Lyapunov-Krasovskii functional, asymptotic stability, Malkin's approach

ABSTRAKT

Dizertační práce se zabývá procesy, které jsou řízeny systémy zpožděných diferenciálních rovnic

$$x'(t) = f(t, x_t, u), \quad t \geq t_0$$

kde $t_0 \in \mathbb{R}$, funkce f je definována v jistém podprostoru množiny $[t_0, \infty) \times C_\tau^m \times \mathbb{R}^r$, $m, r \in \mathbb{N}$, $C_\tau^m = C([-\tau, 0], \mathbb{R}^m)$, $\tau > 0$, $x_t(\theta) := x(t+\theta)$, $\theta \in [-\tau, 0]$, $x\colon [t_0 - \tau, \infty) \to \mathbb{R}^m$. Za předpokladu $f(t, \theta_m^*, \theta_r) = \theta_m$, kde $\theta_m^* \in C_\tau^m$ je nulová vektorová funkce, θ_r a θ_m jsou r a m-dimenzionální nulové vektory, je řídící funkce $u = u(t, x_t)$, $u\colon [t_0, \infty) \times C_\tau^m \to \mathbb{R}^r$, $u(t, \theta_m^*) = \theta_r$ určena tak, že nulové řešení $x(t) = \theta_m$, $t \geq t_0 - \tau$ systému je asymptoticky stabilní a pro libovolné řešení $x = x(t)$ integrál

$$\int_{t_0}^{\infty} \omega\left(t, x_t, u(t, x_t)\right) \mathrm{d}t,$$

kde ω je pozitivně definitní funkcionál, existuje a nabývá své minimální hodnoty v daném smyslu. Pro řešení tohoto problému byla Malkinova metoda pro obyčejné diferenciální systémy rozšířena na zpožděné funkcionální diferenciální rovnice a byla použita druhá metoda Lyapunova. Výsledky jsou ilustrovány příklady a aplikovány na některé třídy zpožděných lineárních diferenciálních rovnic.

KLÍČOVÁ SLOVA

optimální stabilizace, řídící funkce, funkcionál Lyapunova-Krasovského, asymptotická stabilita, Malkinova metoda

DEMCHENKO, Hanna. *Optimization of Delayed Differential Systems by Lyapunov's Direct Method*. Brno, 2018, 89 p. Doctoral thesis. Brno University of Technology, Faculty of Electrical Engineering and Communication, Department of Mathematics. Advised by prof. RNDr. Josef Diblík, DrSc.

DECLARATION

I declare that I have written the Doctoral Thesis titled "Optimization of Delayed Differential Systems by Lyapunov's Direct Method" independently, under the guidance of the advisor and using exclusively the technical references and other sources of information cited in the thesis and listed in the comprehensive bibliography at the end of the thesis.

As the author I furthermore declare that, with respect to the creation of this Doctoral Thesis, I have not infringed any copyright or violated anyone's personal and/or ownership rights. In this context, I am fully aware of the consequences of breaking Regulation § 11 of the Copyright Act No. 121/2000 Coll. of the Czech Republic, as amended, and of any breach of rights related to intellectual property or introduced within amendments to relevant Acts such as the Intellectual Property Act or the Criminal Code, Act No. 40/2009 Coll., Section 2, Head VI, Part 4.

Brno ....

author's signature

ACKNOWLEDGEMENT

I would like to sincerely thank my supervisor prof. RNDr. Josef Diblík, DrSc. for his helpful advice, numerous valuable suggestions, great patience and for his support and guidance during my Ph.D. study.

Brno ....

author's signature

CONTENTS

1 INTRODUCTION

Differential equations are a strong tool for modelling and solving numerous engineering, mechanical, economic or population problems. It is well-known that in such problems a time delay arises quite naturally. For example, in electrical engineering, the time delay can be measured as the difference between the input of a signal in an electrical circuit and its response. In general, there is always a time delay in the real-life processes depending on time. So, differential equations with time delay are an important field of research. As the systems with feedback can be described (under certain conditions) by systems of differential equations with a delay or by difference equations, a wide range of applications is opened for research.

In practical applications, the behaviour of many dynamical systems depends on their previous history. This phenomenon can be brought about by the presence of delays in the equations under consideration. In view of the intrinsic difficulties in solving such problems, progress in this field is slow. This is why using the optimal control of delay systems is so needed and important.

The thesis is devoted to the optimal control problem of delayed differential equations.

The fundamentals of the theory of functional and ordinary differential equations are well described, for example, in books by R.D. Driver [23], J.K. Hale [34], L.E. Elsgolts and S.B. Norkin [24], N.N. Krasovskii [37], R.P. Agarwal, L. Berezansky, E. Braverman and A. Domoshnitsky [3], R.P. Agarwal, M. Bohner and Li Wan-Tong [4], I. Gyori and G. Ladas [33]. Classics in the field of the optimal control are R. Bellman [9], L.S. Pontryagin, V.G. Boltyanskij, R.V. Gamkrelidze and E.F. Mishchenko [52], A.A. Fel'dbaum [25], A.M. Letov [40], [41], V.M. Alekseev, V.M. Tikhomirov and S.V. Fomin [5], I.G. Malkin [45], R. Gabasov and F. Kirillova [29], [30], D.E. Kirk [36], E. Fridman [27], A.V. Kim and A.V. Ivanov [35].

1.1 Current State

Differential equations have been the object of research since the 17th century (after works by Newton and Leibniz) and have been intensively developed for the last century. Monographs summarizing some outcomes were mentioned above in the Introduction.

One of the most important sections of the qualitative theory of functional differential equations is the theory of stability. The method of Lyapunov functionals, proposed by Krasovskii in [37], is still one of the main methods in the research of the delayed system's stability. Analytical research of the stability of some dynamic systems led to the emergence of a new independent field of science - the theory of automatic control (regulation). The basis of this theory is usually associated with the book [47]. The theory of optimal control is an important part in the theory of automatic control, formed primarily on the basis of the classical calculus of variations, the Pontryagin maximum principle [52] and Bellman's dynamic programming [9]. The direct Lyapunov method (Lyapunov function method) [43], which is the basis of the modern nonlinear theory of automatic control, is widely used in modeling control structures of nonlinear systems. In addition to the stability conditions, the method includes an analysis of the quality of control processes.

Numerous papers on the qualitative theory of differential equations, control theory, and optimization are published every year. Some interesting results have been published on representations of solutions of delayed systems [22], [38], on stability of solutions [28], [42], and on optimal control for delayed differential equations [48], [54], [57], [61]. Functional differential equations for modeling the biological problems were first used and investigated in [56]. There are many later works on modeling of biological processes, for example, [1], [2], [31], on applying optimal control in biology and medicine [7], [39], [53]. In [59], the authors introduced a version of the stochastic discrete-time maximum principle for solving an optimal control problem. In [49], the damping of the solution problem is solved by means of a linear difference–differential controller with a state feedback. Here a certain form of the control function was used to stabilize the solution. There are numerous works (for example, [26], [58]) where the authors study the control of systems using some specific control functions.

1.2 Aims of the thesis

The aim of the thesis is to solve the optimal stabilization problem for processes described by a system of delayed differential equations

$$x'(t) = f(t, x_t, u), \quad t \geq t_0,$$

where $t_0 \in \mathbb{R}$, f is defined on a subspace of $[t_0, \infty) \times C_\tau^m \times \mathbb{R}^r$, $m, r \in \mathbb{N}$, $C_\tau^m = C([-\tau, 0], \mathbb{R}^m)$, $\tau > 0$, $x_t(\theta) := x(t+\theta)$, $\theta \in [-\tau, 0]$, $x\colon [t_0 - \tau, \infty) \to \mathbb{R}^m$. Under the assumption $f(t, \theta_m^*, \theta_r) = \theta_m$, where $\theta_m^* \in C_\tau^m$ is a zero vector-function, θ_r and θ_m are r and m-dimensional zero vectors, a control function $u = u(t, x_t)$, $u\colon [t_0, \infty) \times C_\tau^m \to \mathbb{R}^r$, $u(t, \theta_m^*) = \theta_r$ is such that the zero solution $x(t) = \theta_m$, $t \geq t_0 - \tau$ of the system

$$x'(t) = f(t, x_t, u(t, x_t)), \quad t \geq t_0,$$

is asymptotically stable and, for an arbitrary solution $x = x(t)$, the integral

$$\int_{t_0}^{\infty} \omega\left(t, x_t, u(t, x_t)\right) \mathrm{d}t,$$

where ω is a positive-definite functional, exists and attains its minimum value in a given sense.

The thesis solves a problem of optimal stabilization for differential non-delayed and delayed equations and their systems.

The motivation of our research goes back to the results by I.G. Malkin. His book [45] (we refer to the original book written in Russian, to the best of our knowledge, there is no translation into English of the second revised edition, the book [46] is an English translation of the first edition of Malkin's book and does not include the results mentioned) contains, among others, a general principle related to optimal stabilization of ordinary differential systems and its application to linear ordinary differential systems. This principle we apply to some types of linear differential equations and their systems to solve optimal control problems. We analyzed Malkin's approach and, as a result of our investigation, we present its modification to differential delayed systems. Illustrative examples showing how this principle can be applied are developed and, in addition, linear differential delayed systems are considered.

Some results of this work have been already published by the author of the thesis, as a co-author, e.g., in [11]–[19].

1.3 Preliminaries

1.3.1 Stability of functional differential equations

For the auxiliary material given in this part for the reader's convenience, we refer, for example, to [23, 34].
Let $C_\tau^m = C([-\tau, 0], \mathbb{R}^m)$, where $\tau > 0$, $m \in \mathbb{N}$, be the Banach space of continuous mappings $\varphi\colon [-\tau, 0] \to \mathbb{R}^m$. If A is any set in $\mathbb{R}^m$, we will set $C_\tau^m(A) = C([-\tau, 0], A)$.

Let $C_\tau^m(D)$ be the space of continuous mappings from the interval $[-\tau, 0]$ into the set $D = \{\xi \in \mathbb{R}^m\colon \|\xi\| < M\}$, M is a positive constant (or $M = \infty$).

For each $t \geq t_0$, we define $x_t \in C_\tau^m$ by $x_t(\theta) = x(t + \theta)$, $\theta \in [-\tau, 0]$.

Consider a delayed differential system

$$x'(t) = G(t, x_t), \tag{1.1}$$

where $G\colon [\alpha, \infty) \times C_\tau^m(D) \to \mathbb{R}^m$ and $\alpha \in \mathbb{R}$. Given any $t_0 \geq \alpha$ and any $\varphi \in C_\tau^m(D)$, we shall study (1.1) in conjunction with the initial condition

$$x_{t_0} = \varphi. \tag{1.2}$$

Let $x\colon [t_0 - \tau, \infty) \to \mathbb{R}^m$ be a continuous vector-function, $t_0 \in \mathbb{R}$, and let $\tau > 0$ be a number. To emphasize the dependence of x on t_0 and φ, we will sometimes denote $x(t)$ by $x(t; t_0, \varphi)$.
Let $\beta_1 \in \mathbb{R}$, $t_0 < \beta_1 \leq \infty$.

Definition 1.3.1. A continuous function $x\colon [t_0 - \tau, \beta_1) \to D$ with $\beta_1 \in (t_0, \infty)$ is called a solution of the initial problem (1.1), (1.2) on $[t_0 - \tau, \beta_1)$ if the equation (1.1) is satisfied on $[t_0, \beta_1)$ and if $x(t_0 + \theta) = \varphi(\theta)$ for every $\theta \in [-\tau, 0]$.

For a given $t \in [t_0, \infty)$, we define a norm

$$\|x(t)\|_\tau := \max_{\theta \in [-\tau, 0]} (\|x(t + \theta)\|),$$

where

$$\|x(s)\| := \max_{i=1,\dots,n} \{|x_i(s)|\},\ s \in [t_0 - \tau, \infty). \tag{1.3}$$

If $\varphi \in C_\tau^m$ then

$$\|\varphi\|_\tau := \max_{\theta \in [-\tau, 0]} \{\|\varphi(\theta)\|\},$$

where

$$\|\varphi(\theta)\| := \max_{i=1,\dots,m} \{|\varphi_i(\theta)|\}.$$

Let us assume that, for each $t_0 \geq \alpha$, G satisfies the following Condition (C) on $[t_0, \infty) \times C_\tau^m(D)$.

Definition 1.3.2. Condition (C) We say that the functional $G(t, x_t)$ is continuous if it is continuous with respect to t in $[t_0, \infty)$ for each given continuous function $x\colon [t_0 - \tau, \infty) \to \mathbb{R}^m$.

If G satisfies Condition (C), then a continuous function $x\colon [t_0 - \tau, \beta_1) \to D$ is a solution of the initial problem (1.1), (1.2) if and only if

$$x(t) = \begin{cases} \varphi(t - t_0) & \text{for} \quad t_0 - \tau \leq t \leq t_0, \\ \varphi(0) + \displaystyle\int_{t_0}^{t} G(s, x_s) \mathrm{d}s & \text{for} \quad t_0 \leq t < \beta_1. \end{cases} \tag{1.4}$$

Moreover, we will assume that G is locally Lipschitzian and quasi-bounded, see definitions below. Let the symbol J mean either $[t_0, \infty)$ or $[\alpha, \infty)$ as required.

Definition 1.3.3. The functional $G\colon J \times C_\tau^m(D) \to \mathbb{R}^m$ is locally Lipschitzian if, for each given $(t^*, \varphi^*) \in J \times C_\tau^m(D)$, there exist numbers $a > 0$ and $b > 0$ such that

$$\tilde{C} \equiv ([t^* - a, t^* + a] \cap J) \times \{\varphi \in C_\tau^m : \|\varphi - \varphi^*\|_\tau \leq b\}$$

is a subset of $J \times C_\tau^m(D)$ and G is Lipschitzian on $\tilde{C}$. In other words, for some number K (a Lipschitz constant depending on $\tilde{C}$),

$$\|G(t, \varphi) - G(t, \varphi^*)\| \leq K \|\varphi - \varphi^*\|_\tau$$

whenever $(t, \varphi) \in \tilde{C}$ and $(t, \varphi^*) \in \tilde{C}$.

Definition 1.3.4. The functional $G\colon [t_0, \infty) \times C_\tau^m(D) \to \mathbb{R}^m$ is said to be quasi-bounded if G is bounded on every set of the form $[t_0, \beta_1] \times C_\tau^m(A)$, where $t_0 < \beta_1 < \infty$ and A is a closed bounded set of D.

The properties described in Definitions 1.3.2–1.3.4 are basic for ensuring, for example, the existence and uniqueness of a noncontinuable solution of the problem (1.1), (1.2), see Theorem 1.3.5 below, and its continuation (Theorem 1.3.7). The basic theorem on the existence and uniqueness is formulated along with its proof.

Theorem 1.3.5. ***(Local Existence)*** *Let $G\colon [t_0, \infty) \times C_\tau^m(D) \to \mathbb{R}^m$ satisfy Condition (C) and let it be locally Lipschitzian. Then, for each $\varphi \in C_\tau^m(D)$, the initial problem* (1.1), (1.2) *has a unique solution on $[t_0 - \tau, t_0 + \triangle)$ for some $\triangle > 0$.*

Proof. Choose any $a > 0$ and $b > 0$ sufficiently small so that

$$\mathcal{C}^* \equiv [t_0, t_0 + a] \times \{\psi \in C_\tau^m : \|\psi - \varphi\|_\tau \leq b\}$$

is a subset of $[t_0, \infty) \times C_\tau^m(D)$ and G is Lipschitzian on $\mathcal{C}^*$ (say, with a Lipschitz constant K). Define a continuous function $\bar{\chi}$ on $[t_0 - \tau, t_0 + a] \to \mathbb{R}^m$ by

$$\bar{\chi}(t) = \begin{cases} \varphi(t - t_0) & \text{for} \quad t_0 - \tau \leq t \leq t_0, \\ \varphi(0) & \text{for} \quad t_0 \leq t \leq t_0 + a. \end{cases}$$

Then, $G(t, \bar{\chi}_t)$ depends continuously on t and, hence, $\|G(t, \bar{\chi}_t)\| \leq B_1$ on $[t_0, t_0 + a]$ for some constant B_1.
Now define $B = Kb + B_1$. Choose $a_1 \in (0, a]$ such that

$$\|\bar{\chi}_t - \varphi\|_\tau = \|\bar{\chi}_t - \bar{\chi}_{t_0}\|_\tau \leq b \quad \text{for} \quad t_0 \leq t \leq t_0 + a_1.$$

Choose $\Delta > 0$ such that

$$\Delta \leq \min\{a_1, b/B\} \quad \text{and} \quad \Delta < 1/K.$$

Let S be the set of all continuous functions $\chi : [t_0 - \tau, t_0 + \Delta] \to \mathbb{R}^m$ such that

$$\chi(t) = \varphi(t - t_0) \quad \text{for} \quad t_0 - \tau \leq t \leq t_0,$$

and

$$\|\chi(t) - \varphi(0)\| \leq b \quad \text{for} \quad t_0 \leq t \leq t_0 + \Delta.$$

Note that, if $\chi \in S$ and $t \in [t_0, t_0 + \Delta]$, then $\|\chi_t - \bar{\chi}_t\|_r \leq b$; so that

$$\|G(t, \chi_t)\| \leq \|G(t, \chi_t) - G(t, \overline{\chi}_t)\| + \|G(t, \overline{\chi}_t)\| \leq K\|\chi_t - \overline{\chi}_t\| + B_1 \leq B.$$

For each $\chi \in S$, define a function T_χ on $[t_0 - \tau, t^0 + \Delta]$ by

$$(T_\chi)(t) = \begin{cases} \varphi(t - t_0) & \text{for} \quad t_0 - \tau \leq t \leq t_0, \\ \varphi(0) + \displaystyle\int_{t_0}^t G(s, \chi_s) \mathrm{d}s & \text{for} \quad t_0 \leq t \leq t_0 + \Delta. \end{cases}$$

Then, since $\|G(s, \chi_s)\| \leq B$,

$$\|(T_\chi)(t) - \varphi(0)\| \leq B\Delta \leq b \quad \text{for} \quad t_0 \leq t \leq t_0 + \Delta.$$

Also, T_χ is continuous. Thus, $T_\chi \in S$ and we can say that T maps $S \to S$.
Let us now construct "successive approximations" in the usual manner - choosing any $x_{(0)} \in S$ and then defining

$$x_{(1)} = Tx_{(0)}, x_{(2)} = Tx_{(1)}, \ldots.$$

Bear in mind that

$$x_{(\ell)} = \varphi(t - t_0),\ \ell = 0, 1, 2, \dots \quad \text{on} \quad [t_0 - \tau, t_0].$$

Let us prove that the sequence $\{x_{(\ell)}\}$ converges. For each $\ell = 0, 1, 2, \dots,$ when $t_0 \le t \le t_0 + \Delta$:

$$\|x_{(\ell+2)}(t) - x_{(\ell+1)}(t)\| = \left\| \int_{t_0}^{t} \Big[G(s, x_{(\ell+1)s}) - G(s, x_{(\ell)s}) \Big] \,\mathrm{d}s \right\| \le$$
$$K\Delta \cdot \sup_{t_0 \le s \le t_0+\Delta} \|x_{(\ell+1)}(t) - x_{(\ell)}(t)\|.$$

From this and the fact that

$$\|x_{(1)}(t) - x_{(0)}(t)\| \le 2b$$

one finds, for $t_0 \le t \le t_0 + \Delta$,

$$\begin{aligned} \|x_{(2)}(t) - x_{(1)}(t)\| &\le 2bK\Delta, \\ \|x_{(3)}(t) - x_{(2)}(t)\| &\le 2b(K\Delta)^2, \end{aligned}$$

and, by induction

$$\|x_{(\ell+1)}(t) - x_{(\ell)}(t)\| \le 2b(K\Delta)^{\ell}, \quad \ell = 0, 1, 2, \dots.$$

Now, since the series

$$\sum_{p=0}^{\infty} \|x_{(p+1)}(t) - x_{(p)}(t)\| \le \sum_{p=0}^{\infty} 2b(K\Delta)^p$$

converges, the convergence of the sequence $\{x_{(\ell)}\}$ follows by applying the comparison test to each component of

$$x_{(\ell)}(t) = x_{(0)}(t) + \sum_{p=0}^{\ell-1} \Big[x_{(p+1)}(t) - x_{(p)}(t) \Big]$$

on $[t_0, t_0 + \Delta]$. The proof that

$$x(t) \equiv \lim_{\ell \to \infty} x_{(\ell)}(t)$$

satisfies the equation (1.4) is much the same as in the case of ordinary differential equations and we omit it. $\square$

Definition 1.3.6. Let x on $[t_0 - \tau, \beta_1)$ and y on $[t_0 - \tau, \beta_2)$, $\beta_2 > t_0$, both be solutions of the initial problem (1.1), (1.2). If $\beta_2 > \beta_1$, we say y is a continuation of x, or x can be continued to $[t_0 - \tau, \beta_2)$. A solution $x(t)$ of the initial problem (1.1), (1.2) is noncontinuable (on an interval $[t_0 - \tau, \infty)$) if it has no continuation.

Theorem 1.3.7. ***(Extended Existence)*** *Let $G\colon [t_0,\infty) \times C_\tau^m(D) \to \mathbb{R}^m$ satisfy Condition (C) and let it be locally Lipschitzian and quasi-bounded. Then, for each $\varphi \in C_\tau^m(D)$, the problem (1.1), (1.2) has a unique noncontinuable solution x on $[t_0-\tau,\beta_1)$; if $\beta_1 < \infty$, then, for every closed bounded set $A \subset D$, $x(t) \notin A$ for some t in (t_0,β_1).*

Definition 1.3.8. The trivial solution of (1.1) is said to be stable at $t_0 \geq \alpha$ (in the sense of Lyapunov) if, for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that, whenever $\|\varphi\|_\tau < \delta$, the solution $x(t;t_0,\varphi)$ exists on $[t_0-\tau,\infty)$ and

$$\|x(t;t_0,\varphi)\| < \varepsilon$$

for all $t \geq t_0 - \tau$. Otherwise, the trivial solution is said to be unstable at t_0. The trivial solution of (1.1) is said to be uniformly stable on $[\alpha,\infty)$ if it is stable at each $t_0 \geq \alpha$ and the number δ is independent of t_0, i.e., $\delta = \delta(\varepsilon)$ depends only on ε.

Definition 1.3.9. Let $\bar{x}\colon (\alpha-\tau,\infty) \to D$ satisfy the equation (1.1) on $[\alpha,\infty)$. We say that $\bar{x}$ is stable at $t_0 \geq \alpha$ (in the sense of Lyapunov) if, for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon,t_0) > 0$ such that, whenever $\|\varphi - \bar{x}_{t_0}\|_\tau < \delta$, it follows that $x(\cdot;t_0,\varphi)$ exists on $[t_0-\tau,\infty)$ and

$$\|x(t;t_0,\varphi) - \bar{x}(t)\| < \varepsilon$$

for all $t \geq t_0 - \tau$. Otherwise, the solution $\bar{x}$ is said to be unstable at t_0 (in the sense of Lyapunov). The solution $\bar{x}$ of (1.1) is said to be uniformly stable on $[\alpha,\infty)$ if it is stable at each $t_0 \geq \alpha$ and the number δ is independent of t_0, i.e., $\delta = \delta(\varepsilon)$ depends only on ε.

Definition 1.3.10. The trivial solution of (1.1) is said to be uniformly asymptotically stable if it is uniformly stable and there exists a δ_1 (independent of t_0) such that, whenever $t_0 \geq \alpha$ and $\|\varphi\|_\tau < \delta_1$, the expression

$$x(t;t_0,\varphi)$$

tends to 0 as $t \to \infty$ in the following manner:
For each $\eta > 0$, there exists $T = T(\eta) > 0$ (independent of t_0) such that

$$\|x(t;t_0,\varphi)\| < \eta$$

for all $t \geq t_0 + T$.

1.3.2 Lyapunov functionals

The following definitions are related to the estimation of functionals. Throughout the thesis, we will denote by $V = V(t,x_t)$ a functional such that

$$V\colon [t_0,\infty) \times C_\tau^m \to \mathbb{R}. \tag{1.5}$$

Definition 1.3.11. Let a functional V be given. It is called positive-definite if there exists a continuous non-decreasing function $w\colon [0, M) \longrightarrow \mathbb{R}$, $w(0) = 0$, $w(s) > 0$ if $s \in (0, M)$ such that

$$V(t, \psi) \geq w(\|\psi(0)\|) \tag{1.6}$$

on $(\alpha, \infty) \times C_\tau^m(D)$.

Definition 1.3.12. Let a functional V be given. V is said to have an infinitesimal upper bound if there exists a continuous non-decreasing function $W\colon [0, M) \longrightarrow \mathbb{R}$, $W(0) = 0$, $W(s) > 0$ if $s \in (0, M)$ such that

$$V(t, \psi) \leq W(\|\psi\|_\tau) \tag{1.7}$$

on $(\alpha, \infty) \times C_\tau^m(D)$.

Definition 1.3.13. A positive-definite functional V having an infinitesimal upper bound is called a Lyapunov-Krasovskii functional.

To illustrate Definitions 1.3.11, 1.3.12 we set

$$V(t, \psi) = \psi^2(0) + |a| \int_{-r}^{0} \psi^2(\sigma)\, \mathrm{d}\sigma,$$

where $a \neq 0$, $\psi \in C_\tau^m$. For the functional V, the following estimates hold. Since

$$V(t, \psi) = \psi^2(0) + |a| \int_{-\tau}^{0} \psi^2(\sigma)\, \mathrm{d}\sigma \geq \psi^2(0) = \|\psi(0)\|^2 = w(\|\psi(0)\|),$$

where $w(s) = s^2$ and w satisfies all necessary conditions, V is positive-definite. Moreover,

$$\begin{aligned} V(t, \psi) &= \psi^2(0) + |a| \int_{-\tau}^{0} \psi^2(\sigma)\, \mathrm{d}\sigma \\ &\leq \|\psi\|_\tau^2 + |a| \int_{-\tau}^{0} \|\psi\|_\tau^2\, \mathrm{d}\sigma \\ &= (1 + |a|\tau)\|\psi\|_\tau^2 = (1 + |a|\tau) W(\|\psi\|_\tau), \end{aligned}$$

where $W(s) = (1 + |a|\tau)s^2$ and W satisfies all necessary conditions, too. Thus, V has an infinitesimal upper bound.

Definition 1.3.14. Let $x\colon [t_0 - \tau, \infty) \to \mathbb{R}^m$. The derivative of a functional $V(t, x_t)$ at a point $t \geq t_0$ is defined as

$$\frac{\mathrm{d}V(t, x_t)}{\mathrm{d}t} := \lim_{\Delta \to 0} \frac{V(t + \Delta, x_{t+\Delta}) - V(t, x_t)}{\Delta},$$

provided that the limit exists and is finite.

Below we assume that there exists the derivative $\mathrm{d}V(t,x_t)/\mathrm{d}t$ of the functional $V(t,x_t)$ along the trajectories of the differential delayed systems considered, that is, we will assume that x is a solution of a given system.

Theorem 1.3.15. *If there exists a Lyapunov-Krasovskii functional V and if it defines a non-increasing function of t on $[t_0,\infty)$ whenever*

$$x = x(\cdot; t_0, \varphi),\ t \in [t_0 - \tau, \infty)$$

is the noncontinuable solution of (1.1) *through some $(t_0,\varphi) \in [\alpha,\infty) \times C_\tau^m(D)$, then the trivial solution of* (1.1) *is uniformly stable.*

Proof. If, for a given ε, we prove that, for a family of solutions x defined by "small" initial functions, the inequality

$$w(\|x(t)\|) \le w(\varepsilon)$$

holds, then we get

$$\|x(t)\| \le \varepsilon$$

since w is a non-decreasing function. It leads to the stability of a trivial solution. Let $\varepsilon > 0$ be given. Without loss of generality we shall assume $0 < \varepsilon < M$. Then, $w(\varepsilon) > 0$ and, as W is continuous, we can choose $\delta = \delta(\varepsilon) \in (0,\varepsilon)$ such that

$$W(\delta) < w(\varepsilon). \tag{1.8}$$

Now consider any $(t_0,\varphi) \in [\alpha,\infty) \times C_\tau^m(D)$ with $\|\varphi\|_\tau < \delta$. Equation (1.1) has a unique noncontinuable solution $x = x(\cdot; t_0, \varphi)$ through (t_0,φ) on $[t_0 - \tau, \beta_1)$ for some $\beta_1 > t_0$. Thus, using the assumptions of the theorem and condition (1.8), we find for $t_0 \le t < \beta_1$:

$$w(\|x(t)\|) \le V(t,x_t) \le V(t_0,\varphi) \le W(\|\varphi\|_\tau) \le W(\delta) < w(\varepsilon).$$

Now, since w is a non-decreasing function, this can hold only if

$$\|x(t)\| < \varepsilon$$

for $t_0 \le t < \beta_1$. Thus, from Theorem 1.3.7, it follows that $\beta_1 = \infty$ and the assertion of the theorem is proved. $\square$

In the work, we need the following theorem, taken from [23, Theorem C, p. 366].

Theorem 1.3.16. *Let w_1 be a continuous non-decreasing function on $[0,M)$ which is zero at 0 and positive on $(0,M)$. Let $\|G(t,\varphi)\| \le B$ for some constant $B > 0$ for all $(t,\varphi) \in [\alpha,\infty) \times C_\tau^m(D)$. If there exists a Lyapunov-Krasovskii functional V such that, whenever $(t_0,\varphi) \in [\alpha,\infty) \times C_\tau^m(D)$ and $x = x(\cdot; t_0, \varphi)$ on $[t_0 - \tau, \infty)$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} V(t,x_t) \le -w_1(\|x(t)\|) \tag{1.9}$$

for $t \in [t_0,\infty)$, then the trivial solution of (1.1) *is uniformly asymptotically stable.*

Proof. The required uniform stability follows from Theorem 1.3.15. Select and fix some $M_1 \in (0, M)$. Then, choose $\delta_1 > 0$ independent of t_0 such that

$$W(\delta_1) < w(M_1).$$

(It follows that $\delta_1 < M_1$.)
Let an arbitrary $\eta \in (0, \delta_1]$ be given and let $\gamma > 0$ satisfy $W(\gamma) < w(\eta)$. Then, $0 < \gamma < \eta \leq \delta_1$. Choose a positive integer

$$K > \frac{W(\delta_1)}{w_1(\gamma/2)\gamma} \cdot 2B,$$

and define $T(\eta) = K\tau_1$ where $\tau_1 = \max\{\tau, \gamma/B\}$.
Now let $x = x(\cdot; t_0, \varphi)$ be the solution of equation (1.1) through any $(t_0, \varphi) \in [\alpha, \infty) \times C_\tau^m(D)$ with $\|\varphi\|_\tau < \delta_1$. Then, it follows from the uniform stability proof and the choice of δ_1 that x exists and

$$\|x(t)\| \leq M_1 < M$$

for all $t \geq t_0 - \tau$. Now we show that for some $t_1 \in [t_0, t_0 + T(\eta)]$, we have

$$\|x_{t_1}\|_\tau < \gamma.$$

Suppose that, on the contrary,

$$\|x_t\|_\tau \geq \gamma \text{ for all } \ t \in [t_0, t_0 + T(\eta)]. \tag{1.10}$$

From the assumptions of the theorem, we have $\|G(t, \varphi)\| \leq B$ for a $B > 0$ and for all $(t, \varphi) \in [\alpha, \infty) \times C_\tau^m(D)$. It means that

$$\|x'(t)\| = \|G(t, x_t)\| \leq B$$

for every solution $x = x(t)$ such that $x_t \in C_\tau^m(D)$. Since

$$\|x'(t)\| = \max_{i=1,...,m} \{|x_i'(t)|\} \leq B, \ \ t \in [t_0, t_0 + T(\eta)],$$

we have

$$|x_i'(t)| \leq B, \ \ i = 1, ..., m, \ \ t \in [t_0, t_0 + T(\eta)]. \tag{1.11}$$

From (1.11) we get

$$-B \leq x_i'(t) \leq B, \ \ i = 1, ..., m, \ \ t \in [t_0, t_0 + T(\eta)]. \tag{1.12}$$

Integrating (1.12) over an interval $(t_0, t_0 + \varepsilon)$, where $\varepsilon < T(\eta)$, we obtain

$$-B\varepsilon \leq x_i(t + \varepsilon) - x_i(t) \leq B\varepsilon, \ \ i = 1, ..., n, \ \ t \in [t_0, t_0 + T(\eta) - \varepsilon]. \tag{1.13}$$

From (1.10) it follows that there exists an index $i = i^* \in \{1, ..., m\}$ and $t \in [t_0, t_0 + T(\eta) - \varepsilon]$

$$x_{i^*}(t) \geq \gamma$$

and (1.13) gives

$$x_{i^*}(t + \varepsilon) \geq x_{i^*}(t) - B\varepsilon \geq \gamma - B\varepsilon.$$

For $\varepsilon \in [0, \gamma/2B]$ we have

$$x_{i^*}(t + \varepsilon) \geq \frac{\gamma}{2}$$

and we conclude that, on an interval $[t, t + \gamma/2B]$, the inequality

$$\|x(t)\| \geq \frac{\gamma}{2}$$

holds. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t} V(t, x_t) \leq -w_1(\|x(t)\|) \leq -w_1(\gamma/2)$$

and, integrating the last inequality over interval (t_0, t), we derive

$$V(t, x_t) - V(t_0, x_{t_0}) = V(t, x_t) - V(t_0, \varphi) \leq -w_1(\gamma/2)(t - t_0)$$

or

$$V(t, x_t) \leq V(t_0, \varphi) - w_1(\gamma/2)(t - t_0) \leq$$

$$\leq V(t_0, \varphi) - w_1(\gamma/2)T(\eta) = V(t_0, \varphi) - w_1(\gamma/2)K\tau_1.$$

Since $\|\varphi\|_\tau < \delta_1$ and $V(t_0, \varphi) \leq W(\|\varphi\|_\tau) \leq W(\delta_1)$, we get

$$V(t, x_t) \leq V(t_0, \varphi) - Kw_1(\gamma/2)\gamma/(2B) < W(\delta_1) - W(\delta_1) = 0.$$

This inequality contradicts the definition of $V(t, \varphi)$.
From the fact that $\|x_{t_1}\|_\tau < \gamma$ for some $t_1 \in [t_0, t_0 + T(\eta)]$, it follows that, for all $t \geq t_1$,

$$w(\|x(t)\|) \leq V(t, x_t) \leq V(t_1, x_{t_1}) \leq W(\gamma) < w(\eta).$$

Thus, for all $t \geq t_1$, and in particular for all $t \geq t_0 + T(\eta)$,

$$\|x(t)\| < \eta$$

as required for uniform asymptotic stability. $\square$

2 OPTIMIZATION IN NON-DELAYED CASE

In this part, we will investigate a stabilization problem for a system of differential equations without delay. We will be looking for a control function for these systems that satisfies all the desired conditions, such as the best possible quality of a transition process and a minimum value of a quality criterion. A Lyapunov function will be used. Below in parts 2.1 and 2.2 we denote by H a positive number. Parts 2.1 and 2.2 are modifications of parts of [45]. We will use the original concepts and definitions of [45].

2.1 Formulation of the problem

Consider a system of non-delayed functional differential equations

$$x'(t) = F\left(t, x\right), \tag{2.1}$$

where $F\colon \mathfrak{D}_1 \to \mathbb{R}^m$,

$$\mathfrak{D}_1 := \{(t, x) \in [t_0, \infty) \times \mathbb{R}^m,\ \|x\| \leq H\}.$$

Assume that F is continuous and satisfies a local Lipschitz condition with respect to x.
For controllability problems we will consider systems (2.1) with explicitly indicated control functions in the form

$$x'(t) = f\left(t, x, u\right), \tag{2.2}$$

where $f\colon \mathfrak{D} \to \mathbb{R}^m$, $f(t, \Theta_m, \Theta_r) = \Theta_m$,

$$\mathfrak{D} := \{(t, x, u) \in [t_0, \infty) \times \mathbb{R}^m \times \mathbb{R}^r,\ \|x\| \leq H\}.$$

Applied stabilization problems with the requirement of asymptotic stability of a given motion described by the system of differential equations (2.2) require the best possible quality of the transition process. The best quality criterion is very often formulated minimizing the integral

$$I = \int_{t_0}^{\infty} \omega(t, x, u)\mathrm{d}t, \tag{2.3}$$

where $\omega\colon \mathfrak{D} \to \mathbb{R}$. Frequently, the integrand is assumed to have a quadratic form

$$\omega(t, x, u) = x^T C x + u^T D u$$

with a positive-definite constant $m \times m$ matrix C and an $r \times r$ matrix D.

Problem 2.1.1. The optimal control problem is formulated as follows. Find a function $u = u_0$ such that the quality criterion (2.3) is fufilled and the trivial solution of (2.2) is asymptotically stable.
In other words, let a quality criterion of a process $x(t)$ in the form (2.3) be fixed. It is necessary to find a control function $u = u_0$ ensuring the asymptotic stability of non-perturbed motion $x(t) \equiv 0$ such that, for any other admissible control function $u = u^*$, the inequality

$$\int_{t_0}^{\infty} \omega(t, x, u_0) \mathrm{d}t \leq \int_{t_0}^{\infty} \omega(t, x, u^*) \mathrm{d}t$$

holds. The function $u = u_0$ is called an optimal control function.

Definition 2.1.2. Let $V \colon [t_0, \infty) \times \mathbb{R}^m \to [t_0, \infty)$ be a continuous function. Then, V is called a Lyapunov function if it is a locally positive-definite function, i.e.

$$V(t_0, 0) = 0, \;\; V(t_0, x) > 0 \;\text{ for }\; \forall (t, x) \in [t_0, \infty) \times U \backslash \{0\}$$

with U being a neighbourhood region around $x = 0$.

Definition 2.1.3. Let V be a Lyapunov function by Definition 2.1.2. V is said to have an infinitesimal upper bound if there exists a continuous non-decreasing function $W \colon [0, H) \longrightarrow \mathbb{R}$, $W(0) = 0$, $W(s) > 0$ if $s \in (0, H)$ such that

$$V(t, x) \leq W(\|x\|)$$

on $[t_0, \infty) \times \mathbb{R}^m$.

Theorem 2.1.4. *If a function V can be found for the differential equations of the disturbed motion (2.2) satisfying Definition 2.1.2 for which the derivative with respect to time based on these equations* $\mathrm{d}V/\mathrm{d}t$ *is negative and the function V itself permits an infinitesimal upper bound, then the undisturbed motion is asymptotically stable.*

2.2 Malkin's result

Define an auxiliary function $B \colon \mathfrak{D}_2 \to \mathbb{R}$,

$$\mathfrak{D}_2 \colon = \{(v, t, x, u) \in \mathbb{R} \times [t_0, \infty) \times \mathbb{R}^m \times \mathbb{R}^r, \; \|x\| \leq H\},$$

by the formula

$$B\left(V, t, x, u\right) := \frac{\mathrm{d}V(t, x)}{\mathrm{d}t} + \omega(t, x, u), \tag{2.4}$$

where V is a Lyapunov function.
Let us formulate the main theorem of optimal stabilization presented in [45, p. 475–514] utilizing the second Lyapunov method as applied to ordinary differential equations.

Theorem 2.2.1. *Assume that, for the system of differential equations (2.2), there exists a Lyapunov function $V_0(t,x)$ having an infinitesimal upper bound and a function u_0 such that*
i) the function $\omega(t,x,u)$ is positive-definite for every $t \geq t_0$, $\|x\| < H$, $u \in \mathbb{R}^r$;

ii) $B\left(V_0,t,x,u_0\right) \equiv 0$;

iii) $B\left(V_0,t,x,u\right) \geq 0$ for any $u \not\equiv u_0$.

Then, the function u_0 is a solution of the optimal control problem and

$$\int_{t_0}^{\infty} \omega(t,x,u_0)\mathrm{d}t = \min_u \left[\int_{t_0}^{\infty} \omega(t,x,u)\mathrm{d}t\right] = V_0(t_0,x). \tag{2.5}$$

Proof. For $u = u_0$, the function $V_0(t,x)$ satisfies all conditions of the second Lyapunov theorem 2.1.4. For its derivative along the trajectories of the system (2.2), we have (see assumption *ii*) of the theorem)

$$\frac{\mathrm{d}V_0}{\mathrm{d}t} = -\omega(t,x,u_0), \tag{2.6}$$

which means that it is a negative-definite function. That is why, for $u = u_0$, the undisturbed motion $x(t) \equiv 0$ is, by Theorem 2.1.4, asymptotically stable and $\lim_{t\to\infty} x(t) = 0$ for all initial conditions $x(t_0)$ of the region $\|x(t_0)\| \leq \eta$.
Now it is sufficient to show that (2.5) is true. Let a motion $x_0(t)$ satisfy the condition $\|x_0(t)\| \leq h < H$. Obviously, $\eta \leq h$. Thus, during this motion, for all $t \geq t_0$, the equation (2.6) holds. Moreover, from the property of asymptotic stability and, by Definition 2.1.3, we have

$$\lim_{t\to\infty} V_0(t,x_0(t)) = 0. \tag{2.7}$$

Integrating equation (2.6) along the motion $x_0(t)$ over (t_0,∞), using (2.7), we obtain

$$V_0(t_0,x_0(t_0)) = \int_{t_0}^{\infty} \omega(t,x_0(t),u_0)\mathrm{d}t. \tag{2.8}$$

On the other hand, let $u = u_*$ be an arbitrary function that is also a solution of the optimal stabilization problem for the motion $x(t) \equiv 0$ and for initial conditions $\|x(t_0)\| \leq \eta$. Assume that, for $t \geq t_0$, $x_*(t)$ lies inside the region $\|x(t)\| \leq h$. Then, by assumption *iii*), we get

$$\frac{\mathrm{d}V_0}{\mathrm{d}t} \geq -\omega(t,x_*(t),u_*). \tag{2.9}$$

Integrating this inequality over (t_0,∞) and using the property

$$\lim_{t\to\infty} V_0(t,x_*(t)) = 0, \tag{2.10}$$

we obtain

$$V_0(t_0,x_*(t_0)) \leq \int_{t_0}^{\infty} \omega(t,x_*(t),u_*)\mathrm{d}t. \tag{2.11}$$

A similar inequality can be obtained if the motion $x_*(t)$ goes out of the region $\|x(t)\| \leq h$ on an interval. In this case, we have the following situation. Let $\upsilon > t_0$ be the moment of time, when the motion $x_*(t)$ goes back into the region $\|x(t)\| \leq h$ and stays in it for all $t \geq \upsilon$. Then, from that moment on, the equation (2.9) will hold for $x_*(t)$. Integrating this inequality over (υ, ∞) and using the equation (2.10) again, we obtain

$$V_0(\upsilon, x_*(\upsilon)) \leq \int_{\upsilon}^{\infty} \omega(t, x_*(t), u_*)\mathrm{d}t. \tag{2.12}$$

Since $x(t_0)$ satisfies $\|x(t_0)\| \leq \eta$, where η is sufficiently small, we have

$$V_0(t_0, x_*(t_0)) < V_0(\upsilon, x_*(\upsilon)), \tag{2.13}$$

and, due to assumption i), we get

$$\int_{\upsilon}^{\infty} \omega(t, x_*(t), u_*)\mathrm{d}t < \int_{t_0}^{\infty} \omega(t, x_*(t), u_*)\mathrm{d}t. \tag{2.14}$$

From (2.12)–(2.14), we derive (2.11), and from (2.8), (2.11) we get (2.5). □

Remark 2.2.2. The proof of Theorem 2.2.1 is taken from the book [45]. The inequality (2.13) here is not proved in detail. In the generalization for delayed equations, there is a more detailed proof of generalized inequality (see (3.30), (3.31)) from which the inequality (2.13) follows.

2.3 Applications to linear equations and systems

In this part, we apply Theorem 2.2.1 to a class of ordinary differential equations and their systems. The results derived are not included in [45].

2.3.1 Non-delayed equations

Consider a scalar equation

$$x'(t) = ax(t) + bu, \tag{2.15}$$

where a and b ($b \neq 0$) are real constants. Together with the equation (2.15), we will consider the quality criterion (2.3) with

$$\omega(t, x, u) = cx^2(t) + du^2,$$

where $c > 0$, $d > 0$, that is,

$$I = \int_{t_0}^{\infty} (cx^2(t) + du^2)\mathrm{d}t. \tag{2.16}$$

Theorem 2.3.1. *If, for the optimal control problem* (2.15), (2.16), *a Lyapunov function in the form*

$$V(t,x) = hx^2(t),$$

where

$$h = \frac{ad + \sqrt{a^2d^2 + b^2cd}}{b^2}$$

is used, then the optimal control function is

$$u_0 = -\frac{hb}{d}x(t). \tag{2.17}$$

Proof. We need to find a control function $u = u_0$ for which the equation (2.15) with $u = u_0$ is asymptotically stable and a given integral quality criterion (2.16) attains a minimum value. Solving this problem, we need to write an auxiliary functional B by formula (2.4), which should be non-negative due to condition *iii*) of Theorem 2.2.1. We obtain

$$\begin{aligned} B(V,t,x,u) =& 2hx(t)(ax(t) + bu) + cx^2(t) + du^2 \\ =& (2ha + c)x^2(t) + 2hbx(t)u + du^2 \geq 0. \end{aligned}$$

So we need

$$\begin{gathered} 2ha + c \geq 0, \\ (2ha + c)d - h^2b^2 \geq 0. \end{gathered} \tag{2.18}$$

Moreover, $B(V,t,x_t,u_0) \equiv 0$ due to condition *ii*) of Theorem 2.2.1, so

$$du_0^2 + 2hbx(t)u_0 + (2ha + c)x^2(t) \equiv 0$$

and

$$u_0 = -\frac{hb}{d}x(t) \pm \frac{x(t)}{d}\sqrt{h^2b^2 - d(2ha + c)}.$$

For optimal control function existence, the below inequality should hold

$$h^2b^2 - d(2ha + c) \geq 0,$$

which is the opposite of inequality (2.18). So

$$h^2b^2 - d(2ha + c) = 0,$$

and

$$h = \frac{ad + \sqrt{a^2d^2 + b^2cd}}{b^2}.$$

Then,

$$u_0 = -\frac{hb}{d}x(t).$$

□

Remark 2.3.2. System (2.15) with $u = u_0$ given by (2.17) takes the form

$$x'(t) = \left(a - \frac{hb}{d}\right) x(t).$$

Example 2.3.3. Consider the equation (2.15) with $a = -1$, $b = 2$, i.e.,

$$x'(t) = -x(t) + 2u \tag{2.19}$$

with the quadratic quality criterion (2.16), where $c = 1$, $d = 1$, $t_0 = 0$, i.e.,

$$I = \int_0^\infty (x^2(t) + u^2)\mathrm{d}t.$$

By Theorem 2.3.1, an optimal control function is in the form

$$u_0 = \frac{1 - \sqrt{5}}{2} x(t)$$

and the equation (2.15), (2.19) with this control is

$$x'(t) = -\sqrt{5}x(t).$$

By Theorem 2.3.1, the control function for the problem (2.15), (2.16) is given by formula (2.17). In the following example, we demonstrate this statement within a class of control functions.

Example 2.3.4. Consider the equation

$$x'(t) = ax(t) + bu, \tag{2.20}$$

where $b \neq 0$, with the quadratic quality criterion (2.16) with $t_0 = 0$, i.e.,

$$I = \int_0^\infty (cx^2(t) + du^2)\mathrm{d}t. \tag{2.21}$$

An optimal control function by formula (2.17) is in the form

$$u_0 = -\frac{hb}{d} x(t) = -\frac{ad + \sqrt{a^2d^2 + b^2cd}}{bd} x(t). \tag{2.22}$$

After substituting it into equation (2.20), we obtain

$$x'(t) = ax(t) - \frac{ad + \sqrt{a^2d^2 + b^2cd}}{d} x(t),$$

that is,

$$x'(t) = -\frac{1}{d}\sqrt{a^2d^2 + b^2cd}x(t),$$

which has the general solution in the form:

$$x(t) = x(0)e^{-\frac{\sqrt{a^2d^2 + b^2cd}}{d}t}.$$

Using it to find the value of the quality criterion (2.21) with $u = u_0$, we obtain

$$
\begin{aligned}
I =& \int_0^\infty \left(c + \frac{(ad + \sqrt{a^2d^2 + b^2cd})^2}{b^2d} \right) x^2(t)\mathrm{d}t \\
=& \frac{b^2cd + (ad + \sqrt{a^2d^2 + b^2cd})^2}{b^2d} \int_0^\infty x^2(0) e^{-2\frac{\sqrt{a^2d^2+b^2cd}}{d}t} \mathrm{d}t \\
=& x^2(0) \frac{b^2cd + (ad + \sqrt{a^2d^2 + b^2cd})^2}{b^2d} \left(-\frac{de^{-2\frac{\sqrt{a^2d^2+b^2cd}}{d}t}}{2\sqrt{a^2d^2+b^2cd}} \right)\Bigg|_0^\infty \\
=& x^2(0) \frac{b^2cd + (ad + \sqrt{a^2d^2 + b^2cd})^2}{b^2d} \frac{d}{2\sqrt{a^2d^2+b^2cd}} \\
=& x^2(0) \frac{b^2cd + (ad + \sqrt{a^2d^2 + b^2cd})^2}{2b^2\sqrt{a^2d^2+b^2cd}} \\
=& x^2(0) \frac{ad + \sqrt{a^2d^2 + b^2cd}}{b^2} = V(0, x(0)).
\end{aligned} \tag{2.23}
$$

Now we show that (2.22) is the best possible control function in a class of functions. Consider control functions in the form

$$u_\varepsilon = \varepsilon x(t),$$

where ε is a parameter. Again, we substitute it into equation (2.20) to get

$$x'(t) = (a + b\varepsilon)x(t). \tag{2.24}$$

The general solution of the last equation is

$$x(t) = x(0)e^{(a+b\varepsilon)t}.$$

Assume $x(0) \neq 0$ and find the value of the quality criterion with a new control function u_ε.
If $a + b\varepsilon > 0$, then the integral

$$
\begin{aligned}
I =& \int_0^\infty \left(cx^2(t) + du_\varepsilon^2 \right)\mathrm{d}t = \int_0^\infty \left(c + d\varepsilon^2 \right) x^2(t)\mathrm{d}t = \int_0^\infty \left(c + d\varepsilon^2 \right) x^2(0)e^{2(a+b\varepsilon)t}\mathrm{d}t \\
=& \left(c + d\varepsilon^2 \right) x^2(0) \left[\lim_{t\to\infty} \frac{e^{2t(a+b\varepsilon)}}{2(a+b\varepsilon)} - \frac{1}{2(a+b\varepsilon)} \right] = +\infty
\end{aligned}
$$

is obviously divergent (note that $c + d\varepsilon^2 > 0$). If $a + b\varepsilon = 0$, then the integral diverges as well since

$$I = \int_0^\infty \left(c + d\varepsilon^2 \right) x^2(0)\mathrm{d}t = +\infty.$$

For $a + b\varepsilon < 0$, equation (2.24) is asymptotically stable and we get

$$I = -\Big(c + d\varepsilon^2\Big)x^2(0)\frac{1}{2(a+b\varepsilon)}. \tag{2.25}$$

Now we will show that the quality criterion (2.21) attains a minimum value for the optimal control function u_0 defined by (2.22). Comparing (2.23) with (2.25), we need to prove that

$$x^2(0)\frac{ad + \sqrt{a^2d^2 + b^2cd}}{b^2} \le -\Big(c + d\varepsilon^2\Big)x^2(0)\frac{1}{2(a+b\varepsilon)} \tag{2.26}$$

or, after some simplification,

$$\frac{ad + \sqrt{a^2d^2 + b^2cd}}{b^2} \le -\frac{c + d\varepsilon^2}{2(a+b\varepsilon)}. \tag{2.27}$$

Define a function

$$f(\varepsilon) := -\frac{c + d\varepsilon^2}{2(a+b\varepsilon)},$$

and find its minimum. First, the derivative $f'(\varepsilon)$ will be equal to zero if

$$\begin{aligned} f'(\varepsilon) &= -\frac{1}{2(a+b\varepsilon)^2}\Big[2d\varepsilon(a+b\varepsilon) - (c + d\varepsilon^2)b\Big] \\ &= -\frac{1}{2(a+b\varepsilon)^2}\Big[2ad\varepsilon + bd\varepsilon^2 - bc\Big] = 0. \end{aligned} \tag{2.28}$$

The equation (2.28) holds if

$$2ad\varepsilon + bd\varepsilon^2 - bc = 0$$

and the roots ε_1, ε_2 are

$$\varepsilon_{1,2} = \frac{-ad \pm \sqrt{a^2d^2 + b^2cd}}{bd}.$$

The assumption $a + b\varepsilon < 0$, that is in our case,

$$a + b\varepsilon_{1,2} = a + b\frac{-ad \pm \sqrt{a^2d^2 + b^2cd}}{bd} = \pm\frac{1}{d}\sqrt{a^2d^2 + b^2cd} < 0$$

holds only for

$$\varepsilon_2 = \frac{-ad - \sqrt{a^2d^2 + b^2cd}}{bd}.$$

The function $f(\varepsilon_2)$ is equal to the left-hand side of inequality (2.27), since

$$f(\varepsilon_2) = -\frac{c + d\varepsilon_2^2}{2(a+b\varepsilon_2)} = d\frac{c + d\left(\dfrac{-ad - \sqrt{a^2d^2 + b^2cd}}{bd}\right)^2}{2\sqrt{a^2d^2 + b^2cd}}$$

$$= \frac{b^2cd + (ad + \sqrt{a^2d^2 + b^2cd})^2}{2b^2\sqrt{a^2d^2 + b^2cd}} = \frac{2b^2cd + 2a^2d^2 + 2ad\sqrt{a^2d^2 + b^2cd}}{2b^2\sqrt{a^2d^2 + b^2cd}}$$
$$= \frac{\sqrt{a^2d^2 + b^2cd} + ad}{b^2}.$$

To show that it is a minimum value of the function f, we need to find the second derivative

$$f''(\varepsilon_2) = \left[-\frac{1}{2(a + b\varepsilon)^4} \left[(2ad + 2bd\varepsilon)(a + b\varepsilon)^2 - (2ad\varepsilon + bd\varepsilon^2 - bc)2(a + b\varepsilon)b \right] \right] \Bigg|_{\varepsilon=\varepsilon_2}$$
$$= \left[-\frac{d(b\varepsilon + a)}{(a + b\varepsilon)^2} + \frac{b}{(a + b\varepsilon)^3}(2ad\varepsilon + bd\varepsilon^2 - bc) \right] \Bigg|_{\varepsilon=\varepsilon_2} = -\frac{d}{a + b\varepsilon_2}$$

and

$$f''(\varepsilon_2) > 0.$$

So (2.26) holds and (2.22) yields the minimum value of the quality criterion (2.21).

2.3.2 Non-delayed systems with a scalar control function

Consider a linear system with a scalar control function:

$$x'(t) = Ax(t) + bu, \tag{2.29}$$

where $A \in \mathbb{R}^{m \times m}$, $b \in \mathbb{R}^m$, $x(t) \in \mathbb{R}^m$, $u \in \mathbb{R}$. We need to find a control function $u = u_0$ for which the system (2.29) is asymptotically stable and a given integral quality criterion

$$I = \int_{t_0}^{\infty} \left(x^T(t)Cx(t) + du^2 \right) \mathrm{d}t \tag{2.30}$$

has a minimum value provided that C is an $m \times m$ symmetric positive-definite matrix and $d > 0$.
In the sequel, define $\Theta_{\kappa \times \kappa}$ as a zero $\kappa \times \kappa$ matrix.

Theorem 2.3.5. *Assume that there exists a positive-definite symmetric matrix H satisfying the matrix equation*

$$A^T H + HA + C - \frac{1}{d} H b b^T H = \Theta_{m \times m}. \tag{2.31}$$

Then, the optimal stabilization control function $u = u_0$ of the problem (2.29), (2.30) *exists and*

$$u_0 = -\frac{1}{d} b^T H x(t). \tag{2.32}$$

Proof. We will employ Theorem 2.2.1. Define a Lyapunov function

$$V(t,x) = x^T(t)Hx(t),$$

where H is an $m \times m$ positive-definite symmetric matrix. Then, in accordance with the conditions *ii*), *iii*) of Theorem 2.2.1 we analyse the expression B given by (2.4), i.e.,

$$\begin{aligned} B\left(V,t,x,u\right) =& \frac{\mathrm{d}V(t,x)}{\mathrm{d}t} + \omega\left(t,x,u\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(x^T(t)Hx(t)\right) + \omega\left(t,x,u\right) \\ =& [Ax(t)+bu]^T Hx(t) + x^T(t)H[Ax(t)+bu] \\ &+ x^T(t)Cx(t) + du^2. \end{aligned}$$

Simplifying the last expression, we get

$$B\left(V,t,x,u\right) = x^T(t)[A^TH + HA + C]x(t) + 2b^THx(t)u + du^2. \tag{2.33}$$

Looking for an extremum of (2.33) with respect to u, we get

$$B'_u(V,t,x,u) = 2b^THx(t) + 2du = 0,$$

i.e.,

$$u = -\frac{1}{d}b^THx(t), \tag{2.34}$$

which is the minimum of the function B because

$$B''_{uu}(V,t,x,u) = 2d > 0.$$

For $B(V,t,x,u) = 0$ to hold, by (2.33) we have

$$\begin{aligned} &x^T(t)[A^TH + HA + C]x(t) - \frac{1}{d}\left(b^THx(t)\right)^2 \\ &= x^T(t)\left[A^TH + HA + C - \frac{1}{d}Hbb^TH\right]x(t) = 0, \end{aligned}$$

that is,

$$A^TH + HA + C - \frac{1}{d}Hbb^TH = \Theta_{m\times m}.$$

Thus, for the control function defined by (2.34) the desired optimal stabilization control function is

$$u_0 = -\frac{1}{d}b^THx(t).$$

The formula (2.32) is proved. For the control function (2.32) and the Lyapunov function used, the system (2.29) is asymptotically stable and the quality criterion (2.30) takes a minimum value. $\square$

Remark 2.3.6. System (2.29) with $u = u_0$ given by (2.32) takes the form

$$x'(t) = \left(A - \frac{1}{d} b^T H\right) x(t).$$

Example 2.3.7. Let the system (2.29) be reduced to

$$\begin{aligned} x_1'(t) &= -2x_1(t) + x_2(t) + u, \\ x_2'(t) &= x_1(t) - 2x_2(t) + u, \end{aligned} \tag{2.35}$$

with the quality criterion (2.30)

$$I = \int_0^\infty \left(3x_1^2(t) + 3x_2^2(t) + u^2\right) \mathrm{d}t,$$

where

$$A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \ b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ C = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \ d = 1, \ t_0 = 0.$$

By formula (2.32), we obtain the optimal stabilization control function in the form

$$u_0 = -\frac{1}{d} b^T H x(t) = -\begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -(h_1 + h_2)x_1 - (h_2 + h_3)x_2. \tag{2.36}$$

We need to find the matrix H in (2.36). In our case, by (2.31),

$$\begin{aligned} &A^T H + HA + C - \frac{1}{d} H b b^T H \\ &= \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}^T \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix} + \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \\ &\quad - \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix} \\ &= \begin{pmatrix} -4h_1 + 2h_2 + 3 - (h_1 + h_2)^2 & h_1 - 4h_2 + h_3 - (h_1 + h_2)(h_2 + h_3) \\ h_1 - 4h_2 + h_3 - (h_1 + h_2)(h_2 + h_3) & 2h_2 - 4h_3 + 3 - (h_2 + h_3)^2 \end{pmatrix} \\ &= \Theta_{2\times 2}, \end{aligned}$$

which means that

$$\begin{cases} -4h_1 + 2h_2 + 3 - (h_1 + h_2)^2 = 0, & (2.37) \\ h_1 - 4h_2 + h_3 - (h_1 + h_2)(h_2 + h_3) = 0, & (2.38) \\ 2h_2 - 4h_3 + 3 - (h_2 + h_3)^2 = 0. & (2.39) \end{cases}$$

To solve it we can, for example, add the second equation multiplied by 2 to the sum of the first and the third equations ((2.37)+(2.39)+2(2.38)). We obtain

$$-2h_1 - 4h_2 - 2h_3 + 6 - [(h_1 + h_2) + (h_2 + h_3)]^2$$

$$= -2[h_1 + 2h_2 + h_3] + 6 - [h_1 + 2h_2 + h_3]^2 = 0.$$

Denoting

$$h_1 + 2h_2 + h_3 = K, \tag{2.40}$$

we get

$$K^2 + 2K - 6 = 0$$

and $K = -1 \pm \sqrt{7}$.

After subtracting the first equation of the system from the third one, i.e., ((2.39)–(2.37)), we obtain

$$\begin{aligned} 4h_1 - 4h_3 + (h_1 + h_2)^2 - (h_2 + h_3)^2 =& 4(h_1 - h_3) + (h_1 + 2h_2 + h_3)(h_1 - h_3) \\ =& (h_1 - h_3)(4 + K) = 0 \end{aligned}$$

and

$$h_1 = h_3.$$

Using the last equation to (2.40), we find

$$h_1 + h_2 = \frac{K}{2}. \tag{2.41}$$

Next, from (2.38) we obtain

$$2(h_1 - 2h_2) - (h_1 + h_2)^2 = 0 \Rightarrow h_1 - 2h_2 = \frac{K^2}{8}. \tag{2.42}$$

By (2.41) and (2.42), we have

$$h_1 = h_3 = \frac{K}{3} + \frac{K^2}{24},$$

$$h_2 = \frac{K}{6} - \frac{K^2}{24}.$$

For $K = -1 - \sqrt{7}$, the matrix H is not positive-definite so, by (2.36) the optimal stabilization control function is

$$u_0 = \frac{1 - \sqrt{7}}{2}(x_1(t) + x_2(t)).$$

With $u = u_0$, the system (2.35) takes the form

$$\begin{aligned} x_1'(t) =& -\frac{3 + \sqrt{7}}{2} x_1(t) + \frac{3 - \sqrt{7}}{2} x_2(t), \\ x_2'(t) =& \quad \frac{3 - \sqrt{7}}{2} x_1(t) - \frac{3 + \sqrt{7}}{2} x_2(t). \end{aligned}$$

2.3.3 Non-delayed systems with a control vector-function

As the next application consider a system:

$$x'(t) = Ax(t) + Pu, \tag{2.43}$$

where $A \in \mathbb{R}^{m\times m}$, $P \in \mathbb{R}^{m\times r}$, $x(t) \in \mathbb{R}^m$, $u \in \mathbb{R}^r$. We need to find an optimal control function $u = u_0$ for which the system is asymptotically stable and an integral quality criterion

$$I = \int_{t_0}^{\infty} \left(x^T(t)Cx(t) + u^T Du\right) \mathrm{d}t \tag{2.44}$$

takes a minimum value provided that $C \in \mathbb{R}^{m\times m}$ is a symmetric, positive-definite matrix and D is a diagonal control matrix, $D = \operatorname{diag}\{d_j\}$, $d_j > 0$, $j = 1, \dots, r$.

Theorem 2.3.8. *Assume that there exists a positive-definite symmetric matrix H satisfying the matrix equation*

$$A^T H + HA + C - HPD^{-1}P^T H = \Theta_{m\times m}. \tag{2.45}$$

Then, the optimal stabilization control function $u = u_0$ of the problem (2.43), (2.44) *exists and*

$$u_0 = -D^{-1}P^T Hx(t). \tag{2.46}$$

Proof. We will employ Theorem 2.2.1. Define a Lyapunov function

$$V(t, x) = x^T(t)Hx(t),$$

where H is an $m \times m$ positive-definite symmetric matrix. Then, in accordance with the conditions *ii*), *iii*) of Theorem 2.2.1, we analyse the expression B given by (2.4), i.e.,

$$\begin{aligned} B(V, t, x, u) =& \frac{\mathrm{d}V(t, x)}{\mathrm{d}t} + \omega(t, x, u) = \frac{\mathrm{d}}{\mathrm{d}t}\left(x^T(t)Hx(t)\right) + \omega(t, x, u) \\ =& [Ax(t) + Pu]^T Hx(t) + x^T(t)H[Ax(t) + Pu] \\ & + x^T(t)Cx(t) + u^T Du. \end{aligned}$$

Simplifying the last expression, we get

$$\begin{aligned} B(V, t, x, u) =& x^T(t)[A^T H + HA + C]x(t) + u^T(t)P^T Hx(t) \\ & + x^T(t)HPu + u^T Du. \end{aligned} \tag{2.47}$$

Looking for an extremum of (2.47), with respect to u, we get

$$B'_u(V, t, x, u) = 2P^T Hx(t) + 2Du = 0,$$

i.e.,

$$u = -D^{-1}P^T H x(t), \tag{2.48}$$

which is the minimum of the function B because $B''_{uu} = 2D$, and D is a positive-definite matrix.
For $B(V, t, x, u) = 0$ to hold, from (2.47) we get

$$\begin{aligned}
&B(V,t, x, u) \\
&\quad = x^T(t)\Big[A^T H + HA + C - [D^{-1}P^T H]^T P^T H - HPD^{-1}P^T H \\
&\qquad + [D^{-1}P^T H]^T DD^{-1}P^T H\Big]x(t) \\
&\quad = x^T(t)\Big[A^T H + HA + C - HPD^{-1}P^T H - HPD^{-1}P^T H + HPD^{-1}P^T H\Big]x(t) \\
&\quad = x^T(t)\Big[A^T H + HA + C - HPD^{-1}P^T H\Big]x(t) = 0,
\end{aligned}$$

that is,

$$A^T H + HA + C - HPD^{-1}P^T H = \Theta_{m\times m}.$$

From (2.48) and the above computations, we get

$$u_0 = -D^{-1}P^T H x(t).$$

Thus, for the control function (2.46) and the Lyapunov function used, the system (2.43) is asymptotically stable and the quality criterion (2.44) has a minimum value. □

Remark 2.3.9. System (2.43) with $u = u_0$ given by (2.46) takes the form

$$x'(t) = \left(A - PD^{-1}P^T H\right) x(t).$$

Example 2.3.10. Consider the system (2.43) with the quality criterion (2.44). Let the matrices have the form

$$A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \ P = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}, \ C = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \ D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $\varepsilon \neq -1$, so

$$\begin{aligned}
x_1'(t) &= -\,2x_1(t) + \ \ x_2(t) + \ \ u_1 + \varepsilon u_2, \\
x_2'(t) &= \qquad x_1(t) - 2x_2(t) + \varepsilon u_1 + \ \ u_2.
\end{aligned} \tag{2.49}$$

By (2.46) the optimal control function will be in the form

$$u_0 = -D^{-1}P^T H x(t) = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}^T \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$

so

$$
\begin{aligned}
u_1^0 &= -(h_1+\varepsilon h_2)x_1 - (h_2+\varepsilon h_3)x_2,\\
u_2^0 &= -(\varepsilon h_1+h_2)x_1 - (\varepsilon h_2+h_3)x_2. \qquad (2.50)
\end{aligned}
$$

Determine the matrix H. Compute the expression (2.45), i.e.,

$$
\begin{aligned}
&A^T H + HA + C - HPD^{-1}P^T H\\
&= \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}^T \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix} + \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\\
&- \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix} \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}^T \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix} = \Theta_{2\times 2},
\end{aligned}
$$

which means that

$$
\begin{cases}
-4h_1 + 2h_2 + 3 - (h_1+\varepsilon h_2)^2 - (\varepsilon h_1 + h_2)^2 = 0, & (2.51)\\
h_1 - 4h_2 + h_3 - (h_1+\varepsilon h_2)(h_2+\varepsilon h_3) - (\varepsilon h_1+h_2)(\varepsilon h_2+h_3) = 0, & (2.52)\\
2h_2 - 4h_3 + 3 - (h_2+\varepsilon h_3)^2 - (\varepsilon h_2 + h_3)^2 = 0. & (2.53)
\end{cases}
$$

To solve this we can, for example, subtract the first equation from the third one, i.e., ((2.53)–(2.51)) to obtain

$$
\begin{aligned}
&4h_1 - 4h_3 + (h_1+\varepsilon h_2)^2 - (h_2+\varepsilon h_3)^2 + (\varepsilon h_1+h_2)^2 - (\varepsilon h_2+h_3)^2\\
&=4(h_1-h_3) + (h_1+\varepsilon h_2+h_2+\varepsilon h_3)(h_1+\varepsilon h_2-h_2-\varepsilon h_3)\\
&\quad + (\varepsilon h_1+h_2+\varepsilon h_2+h_3)(\varepsilon h_1+h_2-\varepsilon h_2-h_3)\\
&=4(h_1-h_3) + h_2(1+\varepsilon)(h_1-\varepsilon h_3+\varepsilon h_1-h_3)\\
&\quad + (h_1+\varepsilon h_3)(h_1+\varepsilon h_2-h_2-\varepsilon h_3) + (\varepsilon h_1+h_3)(\varepsilon h_1+h_2-\varepsilon h_2-h_3)\\
&=4(h_1-h_3) + h_2(1+\varepsilon)^2(h_1-h_3)\\
&\quad + h_1^2(1+\varepsilon^2) + h_1h_2(2\varepsilon-1-\varepsilon^2) + h_2h_3(\varepsilon^2-2\varepsilon+1) + h_3^2(-\varepsilon^2-1)\\
&=(h_1-h_3)(4+h_2(1+\varepsilon^2)) + (1+\varepsilon^2)(h_1^2-h_3^2) - h_2(\varepsilon-1)^2(h_1-h_3)\\
&=(h_1-h_3)(4+h_2(1+\varepsilon^2) + (h_1+h_3)(1+\varepsilon^2) - h_2(\varepsilon-1)^2)\\
&=(h_1-h_3)(4+2h_2\varepsilon + (h_1+h_3)(1+\varepsilon^2)) = 0.
\end{aligned}
$$

This implies

$$
h_1 = h_3 \qquad (2.54)
$$

since (because the matrix H is positive-definite and $h_1 > |h_2|$)

$$
4 + 2h_2\varepsilon + (h_1+h_3)(1+\varepsilon^2) > 0.
$$

We add the second equation multiplied by 2 to the sum of the first and the third equations ((2.51)+2(2.52)+(2.53)) obtaining

$$-2h_1 - 4h_2 - 2h_3 + 6 - (h_1 + \varepsilon h_2 + h_2 + \varepsilon h_3)^2 - (\varepsilon h_1 + h_2 + \varepsilon h_2 + h_3)^2 = 0.$$

Next, using (2.54), we have

$$-4(h_1 + h_2) + 6 - 2(h_1 + h_2)^2(1 + \varepsilon)^2 = 0.$$

If we put

$$h_1 + h_2 = K > 0, \tag{2.55}$$

then

$$K^2(1 + \varepsilon)^2 + 2K - 3 = 0 \tag{2.56}$$

and

$$K = \frac{-1 + \sqrt{1 + 3(1 + \varepsilon)^2}}{(1 + \varepsilon)^2}. \tag{2.57}$$

From (2.55), we get

$$h_1 = K - h_2 \Rightarrow K > h_2. \tag{2.58}$$

From (2.52), we obtain

$$\begin{aligned} &h_1 - 2h_2 - (h_1 + \varepsilon h_2)(\varepsilon h_1 + h_2) \\ &\quad = K - 3h_2 - (K - h_2 + \varepsilon h_2)(\varepsilon K - \varepsilon h_2 + h_2) = 0. \end{aligned}$$

After simplification, we obtain the following equation

$$h_2^2(\varepsilon - 1)^2 + h_2(-K(\varepsilon - 1)^2 - 3) + K - K^2\varepsilon = 0,$$

where

$$h_2 = \frac{K(\varepsilon - 1)^2 + 3 \pm \sqrt{(K(\varepsilon - 1)^2 + 3)^2 - 4(\varepsilon - 1)^2(K - K^2\varepsilon)}}{2(\varepsilon - 1)^2}$$

and

$$\begin{aligned} &(K(\varepsilon - 1)^2 + 3)^2 - 4(\varepsilon - 1)^2(K - K^2\varepsilon) \\ &\quad = K^2(\varepsilon - 1)^4 + 6K(\varepsilon - 1)^2 + 9 - 4(\varepsilon - 1)^2(K - K^2\varepsilon) \\ &\quad = K^2(\varepsilon - 1)^2((\varepsilon - 1)^2 + 4\varepsilon) + 2K(\varepsilon - 1)^2 + 9 \\ &\quad = K^2(\varepsilon - 1)^2(\varepsilon + 1)^2 + 2K(\varepsilon - 1)^2 + 9 \\ &\quad \overset{\text{by (2.56)}}{=} (\varepsilon - 1)^2(-2K + 3) + 2K(\varepsilon - 1)^2 + 9 = 3(\varepsilon - 1)^2 + 9. \end{aligned}$$

So

$$h_2 = \frac{K(\varepsilon - 1)^2 + 3 \pm \sqrt{3(\varepsilon - 1)^2 + 9}}{2(\varepsilon - 1)^2}. \tag{2.59}$$

For, say, $\varepsilon = 0.5$, from (2.57), (2.59) and (2.58), we obtain

$$K \doteq 0.792837, \quad h_2 \doteq 0.151421, \quad h_1 = h_3 \doteq 0.641416$$

(another solution for $h_2 \doteq 12.6414$ does not satisfy (2.58)).
By (2.50), the optimal stabilization control function will be (coefficients are computed approximately)

$$u_1^0 = -0.7171265 x_1(t) - 0.472129 \;\; x_2(t),$$

$$u_2^0 = -0.472129 \;\; x_1(t) - 0.7171265 x_2(t).$$

The system (2.49) with $u = u_0$ takes the form

$$\begin{aligned} x_1'(t) &= -2.95319 \;\; x_1(t) + 0.169308 x_2(t), \\ x_2'(t) &= \quad 0.169308 x_1(t) - 2.95319 \;\; x_2(t). \end{aligned}$$

3 OPTIMIZATION IN DELAYED CASE

In this part, we will consider systems of delayed scalar equations with constant coefficients. For such equations, we will find control functions theoretically and, in specific examples, by using the formulas obtained. The results of this chapter are new.

3.1 Formulation of the problem

Consider an arbitrary dynamic process and assume that it can be described by a system of functional differential equations of delayed type

$$x'(t) = F\left(t, x_t\right), \tag{3.1}$$

where $F\colon \mathcal{D}_1 \to \mathbb{R}^m$,

$$\mathcal{D}_1 := \{(t, x_t) \in [t_0, \infty) \times C_\tau^m,\ \|x_t\|_\tau \leq M_x\}$$

and M_x is a given positive constant. Let the functional F be continuous, locally Lipschitzian and quasi-bounded. Together with (3.1), we consider the initial problem

$$x_{t_*} = \varphi, \tag{3.2}$$

where $t_* \geq t_0$, and $\varphi \in C_\tau^m$.

Our goal is to be able to control the process. Consider a process $x\colon [t_0-\tau, \infty) \to \mathbb{R}^m$, controlled by means of a control function (or control functional) $u = u(t, x_t)$, where

$$u\colon \mathcal{D}_1 \to \mathbb{R}^r,\ \ u(t, \theta_m^*) = \theta_r$$

such that $\|u(t, x_t)\| \leq M_u$, $(t, x_t) \in \mathcal{D}_1$, M_u is a given positive constant, and assuming that u is continuous, locally Lipschitzian and quasi-bounded. Assume that the process can be modelled by a system of differential equations of delayed type

$$x'(t) = f(t, x_t, u), \quad t \geq t_0, \tag{3.3}$$

where $f\colon \mathcal{D} \to \mathbb{R}^m$,

$$\mathcal{D} := \{(t, x_t, u) \in [t_0, \infty) \times C_\tau^m \times \mathbb{R}^r,\ \|x_t\|_\tau \leq M_x,\ \|u\| \leq M_u\}$$

and $\|u\|$ is defined as in (1.3). Assume that

$$f(t, \theta_m^*, \theta_r) = \theta_m$$

and that f is continuous, locally Lipschitzian and quasi-bounded. Let, moreover, for a constant $K_1 \geq 0$, $\|f(t, x_t, u)\| \leq K_1$ whenever $(t, x_t, u) \in \mathcal{D}$.

If we specify $F(t, x_t) := f(t, x_t, u)$, where $u = u(t, x_t)$, then the system

$$x'(t) = f(t, x_t, u(t, x_t)), \quad t \geq t_0 \tag{3.4}$$

is a particular case of the system (3.1) and (1.1) and, consequently, the auxiliary concepts formulated for (1.1) in part 1.3 can be applied to the system (3.4) as well. In what follows, we will assume, without loss of generality, that the constant M_x is so large that the below solutions of the system (3.4), defined on $[t_0 - \tau, \infty)$, satisfy $\|x(t)\| \leq M_x$, $t \in [t_0 - \tau, \infty)$.
The problem under consideration is formulated as follows.

Problem 3.1.1. Find a control function $u = u_0(t, x_t)$ such that the zero solution $x(t) = \theta_m$, $t \geq t_0 - \tau$ of the system

$$x'(t) = f(t, x_t, u_0(t, x_t)), \quad t \geq t_0, \tag{3.5}$$

is asymptotically stable and, for an arbitrary solution $x\colon [t_0 - \tau, \infty) \to \mathbb{R}^m$ of (3.5), satisfying $\|x_{t_0}\|_\tau \leq \eta$, η is a sufficiently small positive number such that $\eta \leq M_x$, the integral quality criterion

$$I = \int_{t_0}^{\infty} \omega\left(t, x_t, u_0(t, x_t)\right) \mathrm{d}t, \tag{3.6}$$

where $\omega\colon \mathcal{D} \to \mathbb{R}$ is a given positive-definite functional, exists and attains the minimum value. This means that, for an arbitrary control function $u = u^*(t, x_t)$ such that the zero solution $x(t) = \theta_m$, $t \geq t_0 - \tau$ of system

$$x'(t) = f(t, x_t, u^*(t, x_t)), \quad t \geq t_0, \tag{3.7}$$

is asymptotically stable, we have

$$\int_{t_0}^{\infty} \omega\left(t, x_t, u_0(t, x_t)\right) \mathrm{d}t \leq \int_{t_0}^{\infty} \omega\left(t, x_t^*, u^*(t, x_t^*)\right) \mathrm{d}t, \tag{3.8}$$

where $x\colon [t_0 - \tau, \infty) \to \mathbb{R}^m$ is the solution of (3.5) defined by the initial problem (3.2) where $t_* := t_0$, and $x^*\colon [t_0 - \tau, \infty) \to \mathbb{R}^m$ is the solution of (3.7) defined by the same initial problem. The initial function φ in (3.2) is arbitrary except for the assumption $\|\varphi\|_\tau \leq \eta$.

Remark 3.1.2. Modifying the above Definition 1.3.11 of a positive-definite functional to the functional $\omega\colon \mathcal{D} \to \mathbb{R}$ used in (3.6), we specify that ω is a positive-definite functional if there exists a continuous non-decreasing function $w^*(y_1, y_2)$ defined on the set $\mathcal{S} := \{[0, \infty) \times [0, \infty)\}$ such that $w^*(0, 0) = 0$ and $w^*(y_1, y_2) > 0$ if $(y_1, y_2) \in \mathcal{S} \setminus \{(0, 0)\}$, and

$$\omega(t, x_t, u) \geq w^*(\|x(t)\|, \|u\|), \quad t \geq t_0 \tag{3.9}$$

whenever $(t, x_t, u) \in \mathcal{D}$. The non-decreasing property of w^* means that

$$w^*(y_1, y_2) \le w^*(\bar{y}_1, \bar{y}_2) \tag{3.10}$$

whenever $y_1 \le \bar{y}_1$, $y_2 \le \bar{y}_2$ and $(y_1, y_2) \in \mathcal{S}$, $(\bar{y}_1, \bar{y}_2) \in \mathcal{S}$.

Remark 3.1.3. We call the function $u_0(t, x_t)$ solving Problem 3.1.1 the *optimal stabilization control function*. Moreover, the problem of minimizing the integral I by an optimal stabilization control function, as described in Problem 3.1.1, can be formulated more succinctly using the following notation

$$I = \min_u \int_{t_0}^{\infty} \omega(t, x_t, u(t, x_t))\mathrm{d}t.$$

Problem 3.1.1 extends to delayed differential equations Problem II formulated for ordinary differential equations in Malkin's book [45, p. 479]. This problem is formulated above in part 2.1 as well (Problem 2.1.1).

Remark 3.1.4. The optimal stabilization control function $u_0(t, x_t)$, solving Problem 3.1.1, as well as every other control function $u(t, x_t)$ mentioned in the work, is actually a function of the variable t. Therefore, without loss of generality, we sometimes use $u_0(t)$, $u(t)$ or u_0, u for short if there is no danger of ambiguity.

3.2 Generalization of Malkin's result

To solve the problem we are motivated by Malkin's approach, presented in Section 2.2.
Define a functional $B\colon \mathcal{D}_2 \to \mathbb{R}$,

$$\mathcal{D}_2 := \{(v, t, x_t, u) \in \mathbb{R} \times [t_0, \infty) \times C_\tau^m \times \mathbb{R}^r,\ \|x_t\|_\tau \le M_x,\ \|u\| \le M_u\},$$

by the formula

$$B\left(V, t, x_t, u\right) := \frac{\mathrm{d}V(t, x_t)}{\mathrm{d}t} + \omega(t, x_t, u), \tag{3.11}$$

where V is defined by (1.5) and the derivative of V is computed as in Definition 1.3.14 provided that x is an arbitrary fixed solution of the system (3.3).
The next theorem is a generalization of Theorem 2.2.1 for the case of delayed differential equations.

Theorem 3.2.1. *Assume that, for the system of differential equations of delayed type* (3.3)*, there exists a Lyapunov-Krasovskii functional $V(t, x_t)$ and a control function $u_0(t, x_t)$ such that*

i) the functional $\omega\colon \mathcal{D} \to \mathbb{R}$ is positive-definite;

ii) the identity

$$B(V, t, x_t, u_0(t, x_t)) \equiv 0 \tag{3.12}$$

holds on $[t_0, \infty)$ *for every solution* $x\colon [t_0 - \tau, \infty) \to \mathbb{R}^m$ *of the system* (3.3)*, where* $u = u_0(t, x_t)$*;*

iii) the inequality $B(V, t, x_t, u) \geq 0$ *holds on* $[t_0, \infty)$ *for every solution* $x\colon [t_0 - \tau, \infty) \to \mathbb{R}^m$ *of the system* (3.3) *with arbitrary fixed control function* $u = u(t, x_t)$*.*

Then, the function $u_0(t, x_t)$ *is the optimal stabilization control function solving Problem 3.1.1, that is,*

$$I = \min_u \int_{t_0}^{\infty} \omega(t, x_t, u(t, x_t))\mathrm{d}t = \int_{t_0}^{\infty} \omega(t, x_t, u_0(t, x_t))\mathrm{d}t \tag{3.13}$$

and, moreover,

$$\int_{t_0}^{\infty} \omega(t, x_t, u_0(t, x_t))\mathrm{d}t = V(t_0, x_{t_0}). \tag{3.14}$$

Proof. For the derivative $\mathrm{d}V(t, x_t)/\mathrm{d}t$ along the trajectories of the system (3.3) where $u = u_0(t, x_t)$, from (3.11) and (3.12), it follows that

$$\frac{\mathrm{d}V(t, x_t)}{\mathrm{d}t} = -\omega(t, x_t, u_0(t, x_t)), \quad t \geq t_0. \tag{3.15}$$

By (3.9), we have

$$-\omega(t, x_t, u_0(t, x_t)) \leq -w^*(\|x(t)\|, \|u_0(t, x_t)\|), \;\; t \geq t_0.$$

Set

$$w_1(\|x(t)\|) := w^*(\|x(t)\|, 0).$$

Since, by (3.10),

$$w^*(\|x(t)\|, \|u_0(t, x_t)\|) \geq w^*(\|x(t)\|, 0) = w_1(\|x(t)\|),$$

we have

$$\frac{\mathrm{d}V(t, x_t)}{\mathrm{d}t} \leq -w^*(\|x(t)\|, \|u_0(t, x_t)\|) \leq -w_1(\|x(t)\|), \;\; t \geq t_0. \tag{3.16}$$

The functional w_1 is a continuous non-decreasing function on $[0, \infty)$, $w_1(0) = 0$, and $w_1(t) > 0$ for $t \in (0, \infty)$. That is, the functional $V(t, x_t)$ satisfies all the assumptions of Theorem 1.3.16 (the derivative $\mathrm{d}V(t, x_t)/\mathrm{d}t$ satisfies (1.9)).
So, the trivial solution $x(t) \equiv \theta_m$ of the system (3.3) with $u = u_0(t, x_t)$ is uniformly asymptotically stable and there exists an $\eta \in (0, M_x]$ such that, for all initial conditions x_{t_0} satisfying $\|x(t_0)\|_\tau \leq \eta$, the solution $x(t_0, x_{t_0})(t)$ exists on an interval $[t_0 - \tau, \infty)$ and

$$\lim_{t\to\infty} x(t_0, x_{t_0})(t) = \theta_m, \;\; \lim_{t\to\infty} \|(x_0)_t\|_\tau = 0. \tag{3.17}$$

It remains to show that (3.13) and (3.14) are true. Below we will assume that, for a given $h \in (0, M_x)$ the number η is so small that the inequality

$$\sup_{\|x(t_0)\|_\tau \leq \eta} V(t_0, x_{t_0}) < \inf_{\|x(t_0)\|_\tau = h} V(t_0, x_{t_0}) \tag{3.18}$$

is true. Such a choice of the number η is always possible due to the formula (1.6) in Definition 1.3.11 and the formula (1.7) in Definition 1.3.12. Indeed, assuming η so small that $W(\eta) < w(h)$, we have

$$\sup_{\|x(t_0)\|_\tau \leq \eta} V(t_0, x_{t_0}) \leq W(\eta) < w(h) \leq \inf_{\|x(t_0)\|_\tau = h} V(t_0, x_{t_0})$$

and (3.18) holds. Obviously, $\eta < h$. This inequality is a simple consequence of the chain of inequalities

$$W(\eta) < w(h) \leq V(t, h) \leq W(h).$$

Next, we prove that every solution $x^0(t)$, satisfying

$$\left\|x^0(t_0)\right\|_\tau \leq \eta \tag{3.19}$$

satisfies the condition

$$\|x^0(t)\| \leq h < M_x, \quad t \in [t_0, \infty) \tag{3.20}$$

as well. Indeed, due to the properties of w, W and the formula (3.16) saying that the functional $V(t, x_t)$ is non-increasing, we have

$$w(\|x^0(t)|) \leq V(t, x_t^0) \leq V(t_0, x_{t_0}^0) \leq W(\|x_{t_0}^0\|_\tau) \leq W(\eta) < w(h), \tag{3.21}$$

where $t \in [t_0, \infty)$, and inequality

$$w(\|x^0(t)|) < w(h), \quad t \in [t_0, \infty)$$

implies (3.20). Moreover, from the property of asymptotic stability and from (1.7), (3.17), we have

$$0 \leq \lim_{t\to\infty} V(t, x_t^0) \leq \lim_{t\to\infty} W(\left\|x_t^0\right\|_\tau) = 0. \tag{3.22}$$

Set in (3.15) $x := x^0$ where $x = x^0$ is an arbitrary but fixed solution satisfying (3.19). Then, integrating equation (3.15) over an interval (t_0, ∞) and using (3.22), we obtain

$$\begin{aligned}\int_{t_0}^{\infty} \left(\frac{\mathrm{d}V(t, x_t^0)}{\mathrm{d}t}\right) \mathrm{d}t &= \lim_{t\to\infty} V(t, x_t^0) - V(t_0, x_{t_0}^0) \\ &= -V(t_0, x_{t_0}^0) = -\int_{t_0}^{\infty} \omega(t, x_t^0, u_0(t, x_t^0))\mathrm{d}t \end{aligned} \tag{3.23}$$

and

$$V(t_0, x_{t_0}^0) = \int_{t_0}^{\infty} \omega(t, x_t^0, u_0(t, x_t^0))\mathrm{d}t. \tag{3.24}$$

By (3.24), the improper integral exists. Moreover, a consequence of (3.24) is the formula (3.14) as well, provided that u_0 solves Problem 3.1.1.
On the other hand, let $u = u^*$ be an arbitrary control function that is also a solution of Problem 3.1.1. Let $x = x^{**}$ be a solution of the system

$$x'(t) = f(t, x_t, u^*(t, x_t)), \quad t \geq t_0, \tag{3.25}$$

satisfying $\|x^{**}(t_0)\|_\tau \leq \eta$ (recall that the trivial solution of (3.25) is assumed to be asymptotically stable, see the formulation of Problem 3.1.1). Assume

$$\|x^{**}(t)\| \leq h, \ \ t \in [t_0, \infty).$$

By *iii*), we get

$$\begin{aligned} B(V, t, x_t, u) &= B(V(t, x_t^{**}), t, x_t^{**}, u^*(t, x_t^{**})) \\ &= \frac{\mathrm{d}V(t, x_t^{**})}{\mathrm{d}t} + \omega(t, x_t^{**}, u^*(t, x_t^{**})) \geq 0, \ \ t \in [t_0, \infty) \end{aligned}$$

or, equivalently,

$$\frac{\mathrm{d}V(t, x_t^{**})}{\mathrm{d}t} \geq -\omega(t, x_t^{**}, u^*(t, x_t^{**})), \ \ t \in [t_0, \infty). \tag{3.26}$$

Integrating this inequality over (t_0, ∞) and using the property

$$\lim_{t \to \infty} V(t, x_t^{**}) = 0, \tag{3.27}$$

deduced from (1.7), we obtain (computations are similar to those in (3.23))

$$V(t_0, x_{t_0}^{**}) \leq \int_{t_0}^{\infty} \omega(t, x_t^{**}, u^*(t, x_t^{**}))\mathrm{d}t. \tag{3.28}$$

We show that the inequality (3.28) holds even in the case of the solution $x^{**}(t)$ being out of the domain $\|x\| \leq h$ on an interval (because of asymptotic stability, this interval is finite). Assume such a behaviour. Let $t_1 > t_0$ be the moment of time, at which $x^{**}(t)$ goes back into the domain $\|x\| \leq h$ and stays in it for all $t \geq t_1$. Then, from that moment on, the inequality (3.26) will hold for $x^{**}(t)$. Integrating this inequality over (t_1, ∞) and using the property (3.27) again, we obtain

$$V(t_1, x_{t_1}^{**}) \leq \int_{t_1}^{\infty} \omega(t, x_t^{**}, u^*(t, x_t^{**}))\mathrm{d}t. \tag{3.29}$$

Since x^{**} satisfies $\|x^{**}(t_0)\|_\tau \leq \eta$ and $\|x^{**}(t_1)\| = h$, we have (estimates are derived in much the same way as in (3.21))

$$V(t_0, x_{t_0}^{**}) \leq W(\|x_{t_0}^{**}\|_\tau) \leq W(\eta) < w(h) = w(\|x^{**}(t_1)\|) \leq V(t_1, x_{t_1}^{**}), \tag{3.30}$$

that is,

$$V(t_0, x_{t_0}^{**}) < V(t_1, x_{t_1}^{**}). \tag{3.31}$$

Due to the positive-definiteness of ω (assumption i)), we have

$$\int_{t_1}^{\infty} \omega(t, x_t^{**}, u^*(t, x_t^{**}))\mathrm{d}t \le \int_{t_0}^{\infty} \omega(t, x_t^{**}, u^*(t, x_t^{**}))\mathrm{d}t. \tag{3.32}$$

Now, from (3.29)–(3.32) it follows

$$V(t_0, x_{t_0}^{**}) < V(t_1, x_{t_1}^{**}) \le \int_{t_1}^{\infty} \omega(t, x_t^{**}, u^*(t, x_t^{**}))\mathrm{d}t \le \int_{t_0}^{\infty} \omega(t, x_t^{**}, u^*(t, x_t^{**}))\mathrm{d}t$$

and (3.28) holds again.
Finally, assume that the initial function $x_{t_0}^0$ (used, among others, on the left-hand side of (3.24)) and the initial function $x_{t_0}^{**}$ (used, among others, on the left-hand side of (3.28)) are identical, that is $x^0(t_0+\theta) = x^{**}(t_0+\theta)$, $\theta \in [-\tau, 0]$. Then, (3.24) and (3.28) imply

$$\int_{t_0}^{\infty} \omega(t, x_t^0, u_0(t, x_t^0))\mathrm{d}t = V(t_0, x_{t_0}^0) = V(t_0, x_{t_0}^{**}) \le \int_{t_0}^{\infty} \omega(t, x_t^{**}, u^*(t, x_t^{**}))\mathrm{d}t,$$

that is, (3.8) holds with

$$x_t := x_t^0, \;\; u_0(t, x_t) := u_0(t, x_t^0)$$

on the left-hand side and with

$$x_t^* := x_t^{**}, \;\; u^*(t, x_t^*) := u^*(t, x_t^{**}))$$

on the right-hand side. Therefore, the optimal stabilization control function $u = u_0(t, x_t) \equiv u_0(t, x_t^0)$ solves Problem 3.1.1 and (3.13) holds. □

Remark 3.2.2. Theorem 3.2.1 is an extension to delayed differential equations of Theorem IV in Malkin's book [45, p. 485] formulated there for ordinary differential equations. Optimal problems for delayed differential equations with integral quality criteria are often considered for a finite upper limit in an integral quality criterion I and are, in general, not applicable to the case of this limit being infinite (we refer, for example, to [6, 8, 9, 29, 30, 36, 37, 44, 50, 51, 55, 60] and to the references therein). In [27, 35], the quality criteria are considered in an integral form with an infinite upper limit. Unlike our investigation, a control law is searching in the prescribed class of functionals. In [10], an integral quality criterion with an infinite upper limit is used for solving an optimal control problem, but a weight function of an exponential type is used to preserve its convergence.

3.3 Examples

This part uses four examples to illustrate Theorem 3.2.1 with the Lyapunov-Krasovskii functional being chosen in the form

$$V(t, x_t) = hx^2(t) + d\int_{t-\tau}^{t} x^2(s)\mathrm{d}s$$

and

$$h > 0, \quad d > 0. \tag{3.33}$$

The first example shows that the method can be applied to nonlinear equations.

Example 3.3.1. Let $m = r = 1$. Let the equation (3.3) be reduced to a nonlinear delayed equation

$$x'(t) = f(t, x_t, u) := ax(t) + bx(t-\tau) + cx^3(t) + ex(t)u, \tag{3.34}$$

where a, b, c and e are real constants, $\tau > 0$ is a delay and u is a control function. Solve the problem of minimizing I, where

$$\omega\left(t, x_t, u\right) := \alpha x^2(t) + 2\beta x(t)x(t-\tau) + \gamma x^2(t-\tau) + \delta u^2$$

and assume

$$\alpha > 0, \;\; \alpha\gamma - \beta^2 > 0, \;\; \delta > 0. \tag{3.35}$$

Then, ω is a positive-definite functional. The functional B, defined by (3.11), equals

$$\begin{aligned}
B\left(V, t, x_t, u\right) =& \frac{\mathrm{d}V(t, x_t)}{\mathrm{d}t} + \omega\left(t, x_t, u\right)\\
=& \frac{\mathrm{d}}{\mathrm{d}t}\left(hx^2(t) + d\int_{t-\tau}^{t} x^2(s)\mathrm{d}s\right) + \omega\left(t, x_t, u\right)\\
=& 2hx(t)[ax(t) + bx(t-\tau) + cx^3(t) + ex(t)u] + d[x^2(t) - x^2(t-\tau)]\\
&+ \alpha x^2(t) + 2\beta x(t)x(t-\tau) + \gamma x^2(t-\tau) + \delta u^2\\
=& 2hcx^4(t) + (2ha + d + \alpha)x^2(t) + (\gamma - d)x^2(t-\tau)\\
&+ (2hb + 2\beta)x(t)x(t-\tau) + 2hex^2(t)u + \delta u^2.
\end{aligned}$$

To satisfy conditions *ii*) and *iii*) we look for an extremum of B with respect to u. We get

$$B'_u(V, t, x_t, u) = 2hex^2(t) + 2\delta u$$

and the derivative equals zero if

$$u = u_0 = -\frac{hex^2(t)}{\delta}\,. \tag{3.36}$$

Since $B''_{uu}(V,t,x_t,u) = 2\delta > 0$, the value (3.36) is a unique point of minimum. In accordance with (3.12), it is necessary that $B \equiv 0$ if the control function is defined by (3.36), therefore, the following must hold

$$B\left(V,t,x_t,u_0\right) = \left(2hc - \frac{h^2e^2}{\delta}\right)x^4(t) + (2ha+d+\alpha)x^2(t) \\ + (\gamma-d)x^2(t-\tau) + (2hb+2\beta)x(t)x(t-\tau) \equiv 0.$$

This is possible if

$$2c - \frac{he^2}{\delta} = 0, \tag{3.37}$$

$$2ha+d+\alpha = 0, \tag{3.38}$$

$$d = \gamma, \tag{3.39}$$

$$h = -\frac{\beta}{b}. \tag{3.40}$$

From the above consideration, it follows that *iii*) holds as well. If conditions (3.37)–(3.40) together with (3.33) and (3.35) are fulfilled, Theorem 3.2.1 can be applied. Therefore, u_0 defined by the formula (3.36) is the desired optimal control function and the equation (3.34) takes the form

$$x'(t) = f(t,x_t,-hex^2(t)/\delta) := ax(t) + bx(t-\tau) - cx^3(t). \tag{3.41}$$

The coefficient conditions (3.33), (3.35), (3.37)–(3.40) are fulfilled, for example, for the choice

$$a=-2, \quad b=\gamma=\delta=d=h=1, \quad c=e=2, \quad \alpha=3, \quad \beta=-1.$$

Then, $u_0 = -2x^2(t)$ and equations (3.34), (3.41) take the form

$$x'(t) = f(t,x_t,-2x^2(t)) := -2x(t) + x(t-\tau) - 2x^3(t).$$

Remark 3.3.2. The above computations are applicable to some classes of equations with variable coefficients. Consider, for example, the equation

$$x'(t) = f(t,x_t,u) := (a+1/t^2)x(t) + bx(t-\tau) + cx^3(t) + ex(t)u,$$

where the coefficient a is perturbed by a small function (assuming that $t \geq t_0$ and t_0 is sufficiently large). The problem of minimizing I, where

$$\omega\left(t,x_t,u\right) := (\alpha - 2/t^2)x^2(t) + 2\beta x(t)x(t-\tau) + \gamma x^2(t-\tau) + \delta u^2,$$

is solvable with the same control function.

Example 3.3.3. Let $m = r = 1$. Let the equation (3.3) be reduced to a nonlinear delayed equation

$$x'(t) = f(t, x_t, u) := ax(t) + bx(t-\tau) + cx(t)x^2(t-\tau) + ex(t-\tau)u, \tag{3.42}$$

where a, b, c and e are real constants, $\tau > 0$ is a delay and u is a control function. Solve the problem of minimizing I, where

$$\omega(t, x_t, u) := \alpha x^2(t) + 2\beta x(t)x(t-\tau) + \gamma x^2(t-\tau) + \delta u^2,$$

and assume

$$\alpha > 0, \;\; \alpha\gamma - \beta^2 > 0, \;\; \delta > 0. \tag{3.43}$$

Then, ω is a positive-definite functional. The functional B, defined by (3.11), equals

$$\begin{aligned}
B(V, t, x_t, u) =& \frac{\mathrm{d}V(t, x_t)}{\mathrm{d}t} + \omega(t, x_t, u) \\
=& \frac{\mathrm{d}}{\mathrm{d}t}\left(hx^2(t) + d\int_{t-\tau}^{t} x^2(s)\mathrm{d}s\right) + \omega(t, x_t, u) \\
=& 2hx(t)[ax(t) + bx(t-\tau) + cx(t)x^2(t-\tau) + ex(t-\tau)u] \\
&+ d[x^2(t) - x^2(t-\tau)] + \alpha x^2(t) + 2\beta x(t)x(t-\tau) + \gamma x^2(t-\tau) + \delta u^2 \\
=& 2hcx^2(t)x^2(t-\tau) + (2ha + d + \alpha)x^2(t) + (\gamma - d)x^2(t-\tau) \\
&+ (2hb + 2\beta)x(t)x(t-\tau) + 2hex(t)x(t-\tau)u + \delta u^2.
\end{aligned}$$

To satisfy conditions *ii*) and *iii*) we look for an extremum of B with respect to u. We get

$$B'_u(V, t, x_t, u) = 2hex(t)x(t-\tau) + 2\delta u$$

and the derivative equals zero if

$$u = u_0 = -\frac{hex(t)x(t-\tau)}{\delta}\,. \tag{3.44}$$

Since

$$B''_{uu}(V, t, x_t, u) = 2\delta > 0,$$

(3.44) is a unique point of minimum.
In accordance with (3.12), it is necessary that $B \equiv 0$ if the control function is defined by (3.44), therefore, the following must hold

$$\begin{aligned}
B(V, t, x_t, u_0) = &\left(2hc - \frac{h^2e^2}{\delta}\right)x^2(t)x^2(t-\tau) + (2ha + d + \alpha)x^2(t) \\
&+ (\gamma - d)x^2(t-\tau) + (2hb + 2\beta)x(t)x(t-\tau) \equiv 0.
\end{aligned}$$

This is possible if

$$\begin{aligned} 2c - \frac{he^2}{\delta} = & \quad 0, & (3.45) \\ 2ha + d + \alpha = & \quad 0, & (3.46) \\ d = & \quad \gamma, & (3.47) \\ h = & -\frac{\beta}{b}. & (3.48) \end{aligned}$$

From the above consideration, it follows that $iii)$ holds as well. If conditions (3.45)–(3.48) together with (3.33) and (3.43) are fulfilled, Theorem 3.2.1 can be applied. Therefore, u_0 defined by formula (3.44) is the desired optimal control function and equation (3.42) takes the form

$$x'(t) = f(t, x_t, -hex(t)x(t-\tau)/\delta) := ax(t) + bx(t-\tau) - cx(t)x^2(t-\tau). \quad (3.49)$$

The coefficient conditions (3.33), (3.43), (3.45)–(3.48) are fulfilled, for example, for the choice

$$\begin{aligned} a = & -2, \\ b = \gamma = \delta = d = h = & \quad 1, \\ c = e = & \quad 2, \\ \alpha = & \quad 3, \\ \beta = & -1. \end{aligned}$$

Then, $u_0 = -2x(t)x(t-\tau)$ and equations (3.42), (3.49) take the form

$$x'(t) = f(t, x_t, -2x(t)x(t-\tau)) := -2x(t) + x(t-\tau) - 2x(t)x^2(t-\tau).$$

By the following example, which is sort of a generalization of Example 3.3.1, we demonstrate the variability of the method if the control functions affect both the nonlinear and linear terms.

Example 3.3.4. Let $m = 1$, $r = 2$. Let a nonlinear delayed equation of the type (3.3) be of the form

$$x'(t) = f(t, x_t, u) := ax(t) + bx(t-\tau) + cx^3(t) + e_1 x(t) u_1 + e_2 u_2, \quad (3.50)$$

where a, b, c, e_1 and e_2 are real constants, $\tau > 0$ is a delay and u_1, u_2 are control functions. We will solve the problem of minimizing I, where

$$\omega(t, x_t, u) := \alpha x^2(t) + 2\beta x(t)x(t-\tau) + \gamma x^2(t-\tau) + \delta_1 u_1^2 + \delta_2 u_2^2,$$

and assume

$$\alpha > 0, \;\; \alpha\gamma - \beta^2 > 0, \;\; \delta_1 > 0, \;\; \delta_2 > 0. \quad (3.51)$$

Obviously, ω is positive-definite. The functional B defined by (3.11) equals

$$\begin{aligned}
B\left(V,t,x_t,u\right) =& \frac{\mathrm{d}V(t,x_t)}{\mathrm{d}t} + \omega\left(t,x_t,u\right) \\
=& \frac{\mathrm{d}}{\mathrm{d}t}\left(hx^2(t) + d\int_{t-\tau}^{t} x^2(s)\mathrm{d}s\right) + \omega\left(t,x_t,u\right) \\
=& 2hx(t)[ax(t) + bx(t-\tau) + cx^3(t) + e_1x(t)u_1 + e_2u_2] \\
& + d[x^2(t) - x^2(t-\tau)] + \alpha x^2(t) + 2\beta x(t)x(t-\tau) + \gamma x^2(t-\tau) \\
& + \delta_1 u_1^2 + \delta_2 u_2^2 \\
=& 2hcx^4(t) + (2ha + d + \alpha)x^2(t) + (\gamma - d)x^2(t-\tau) \\
& + (2hb + 2\beta)x(t)x(t-\tau) + 2he_1x^2(t)u_1 + 2he_2x(t)u_2 \\
& + \delta_1 u_1^2 + \delta_2 u_2^2.
\end{aligned}$$

To satisfy conditions *ii*) and *iii*) we look for an extremum of B with respect to u_1, u_2. We get

$$\begin{aligned}
B'_{u_1}(V,t,x_t,u) &= 2he_1x^2(t) + 2\delta_1 u_1, \\
B'_{u_2}(V,t,x_t,u) &= 2he_2x(t) + 2\delta_2 u_2.
\end{aligned}$$

The partial derivatives equal zero if

$$u_1 = u_{10} = -\frac{he_1x^2(t)}{\delta_1}\,, \tag{3.52}$$

$$u_2 = u_{20} = -\frac{he_2x(t)}{\delta_2}\,. \tag{3.53}$$

Since $B''_{u_1u_1}(V,t,x_t,u) = 2\delta_1 > 0$, $B''_{u_2u_2}(V,t,x_t,u) = 2\delta_2 > 0$, $B''_{u_1u_2}(V,t,x_t,u) = 0$, the values (3.52), (3.53) determine a unique point of minimum. In accordance with (3.12), it is necessary that $B \equiv 0$ if control functions are defined by (3.52), (3.53), so the following must hold

$$\begin{aligned}
B\left(V,t,x_t,u_0\right) = \left(2hc - \frac{h^2e_1^2}{\delta_1}\right)x^4(t) + \left(2ha + d + \alpha - \frac{h^2e_2^2}{\delta_2}\right)x^2(t) \\
+ (\gamma - d)x^2(t-\tau) + (2hb + 2\beta)x(t)x(t-\tau) \equiv 0.
\end{aligned}$$

This is possible if

$$2c - \frac{he_1^2}{\delta_1} = 0, \tag{3.54}$$

$$2ha + d + \alpha - \frac{h^2e_2^2}{\delta_2} = 0, \tag{3.55}$$

$$d = \quad \gamma, \tag{3.56}$$

$$h = -\frac{\beta}{b}. \tag{3.57}$$

From the above consideration, it follows that *iii*) holds as well. If (3.54)–(3.57) and (3.33), (3.51) are also fulfilled, then Theorem 3.2.1 holds. Therefore, u_1, u_2 defined by formulas (3.52), (3.53) are optimal control functions and the equation (3.50) takes the form

$$x'(t) = f(t, x_t, u_1, u_2) := \left(a - \frac{he_2^2}{\delta_2}\right) x(t) + bx(t-\tau) - cx^3(t). \tag{3.58}$$

The coefficient conditions (3.33), (3.51), (3.54)–(3.57) are fulfilled, for example, for the choice

$$a = -2, \quad b = \gamma = \delta_1 = \delta_2 = e_2 = d = h = 1, \quad c = e_1 = 2, \quad \alpha = 4, \quad \beta = -1.$$

Then, $u_1 = -2x^2(t)$, $u_2 = -x(t)$ and equations (3.50), (3.58) take the form

$$x'(t) = f(t, x_t, -2x^2(t), -x(t)) := -3x(t) + x(t-\tau) - 2x^3(t).$$

Remark 3.3.5. The coefficient $(a - he_2^2/\delta_2)$ in (3.58) is always negative. This is obvious for $a \le 0$. Let $a > 0$. Then,

$$a - \frac{he_2^2}{\delta_2} = \frac{1}{h}\left(ha - \frac{h^2e_2^2}{\delta_2}\right) \overset{\text{by (3.55)}}{=} \frac{1}{h}(-ha - d - \alpha) < 0.$$

In the last example, we show that the control function can depend on the solution with delayed argument and this dependence is caused by the form of ω.

Example 3.3.6. Let $m = r = 1$. Consider a delayed equation (3.3) of the form

$$x'(t) = f(t, x_t, u) := ax(t) + bx(t-\tau) + cu, \tag{3.59}$$

where a, $b \neq 0$ and c are real constants, $\tau > 0$ is a delay and u is a control function. Let ω in I be defined as

$$\begin{aligned} \omega\left(t, x_t, u\right) := \alpha x^2(t) + 2\beta x(t)x(t-\tau) + \gamma x^2(t-\tau) \\ + 2\delta x(t)u + 2\varepsilon x(t-\tau)u + \xi u^2 \end{aligned} \tag{3.60}$$

where α, β, γ, δ, $\varepsilon \neq 0$ and ξ are real constants. Assume

$$\alpha > 0, \quad \begin{vmatrix} \alpha & \beta \\ \beta & \gamma \end{vmatrix} > 0, \quad \begin{vmatrix} \alpha & \beta & \delta \\ \beta & \gamma & \varepsilon \\ \delta & \varepsilon & \xi \end{vmatrix} > 0. \tag{3.61}$$

Then, ω is positive-definite. The functional B defined by (3.11) equals

$$\begin{aligned}
B\left(V,t,x_t,u\right) =& \frac{\mathrm{d}V(t,x_t)}{\mathrm{d}t} + \omega\left(t,x_t,u\right) \\
=& \frac{\mathrm{d}}{\mathrm{d}t}\left(hx^2(t) + d\int_{t-\tau}^{t} x^2(s)\mathrm{d}s\right) + \omega\left(t,x_t,u\right) \\
=& 2hx(t)[ax(t)+bx(t-\tau)+cu(t)] + d[x^2(t)-x^2(t-\tau)] \\
&+ \alpha x^2(t) + 2\beta x(t)x(t-\tau) + \gamma x^2(t-\tau) + 2\delta x(t)u \\
&+ 2\varepsilon x(t-\tau)u + \xi u^2 \\
=& (2ha+d+\alpha)x^2(t) + (2hb+2\beta)x(t)x(t-\tau) + (\gamma-d)x^2(t-\tau) \\
&+ (2hc+2\delta)x(t)u + 2\varepsilon x(t-\tau)u + \xi u^2.
\end{aligned}$$

To satisfy conditions *ii*) and *iii*) we look for an extremum of B with respect to u. We get

$$B'_u(V,t,x_t,u) = (2hc+2\delta)x(t) + 2\varepsilon x(t-\tau) + 2\xi u$$

and the derivative equals zero if

$$u = u_0 = -\frac{1}{\xi}((hc+\delta)x(t) + \varepsilon x(t-\tau)). \tag{3.62}$$

Since $B''_{uu}(V,t,x_t,u) = 2\xi > 0$, the value (3.62) is a unique point of minimum (the positivity of ξ follows from (3.61)). In accordance with (3.12), the value of B for the control function u_0 defined by (3.62) equals zero. Since

$$\begin{aligned}
(2hc+2\delta)x(t)u_0 =& -\frac{1}{\xi}(2hc+2\delta)x(t)((hc+\delta)x(t)+\varepsilon x(t-\tau)) \\
=& -\frac{2}{\xi}(hc+\delta)^2x^2(t) - \frac{2\varepsilon}{\xi}(hc+\delta)x(t)x(t-\tau), \\
2\varepsilon x(t-\tau)u_0 =& -\frac{2\varepsilon}{\xi}x(t-\tau)((hc+\delta)x(t)+\varepsilon x(t-\tau)) \\
=& -\frac{2\varepsilon}{\xi}(hc+\delta)x(t)x(t-\tau) - \frac{2\varepsilon^2}{\xi}x^2(t-\tau), \\
\xi u_0^2 =& \xi\left(\frac{1}{\xi}((hc+\delta)x(t)+\varepsilon x(t-\tau))\right)^2 \\
=& \frac{1}{\xi}(hc+\delta)^2x^2(t) + \frac{2\varepsilon}{\xi}(hc+\delta)x(t)x(t-\tau) + \frac{\varepsilon^2}{\xi}x^2(t-\tau),
\end{aligned}$$

the following must hold

$$B\left(V,t,x_t,u_0\right) = \left(2ha+d+\alpha-\frac{(hc+\delta)^2}{\xi}\right)x^2(t) + \left(\gamma-d-\frac{\varepsilon^2}{\xi}\right)x^2(t-\tau)$$

$$+\left(2hb + 2\beta - \frac{2\varepsilon(hc+\delta)}{\xi}\right) x(t)x(t-\tau) \equiv 0$$

and, therefore,

$$\begin{aligned} 2ha + d + \alpha - \frac{(hc+\delta)^2}{\xi} &= 0, \\ \gamma - d - \frac{\varepsilon^2}{\xi} &= 0, \\ hb + \beta - \frac{\varepsilon(hc+\delta)}{\xi} &= 0. \end{aligned}$$

So the coefficient d could be found from the second equation as

$$d = \gamma - \frac{\varepsilon^2}{\xi}. \tag{3.63}$$

The remaining two equations can be transformed as follows

$$2ha + d + \alpha - \frac{\xi(hb+\beta)^2}{\varepsilon^2} = 0, \tag{3.64}$$

$$(hb+\beta)^2 = \frac{\varepsilon^2(hc+\delta)^2}{\xi^2}. \tag{3.65}$$

Rewrite the equation (3.64) as

$$2ha\varepsilon^2 + d\varepsilon^2 + \alpha\varepsilon^2 - h^2b^2\xi - 2hb\beta\xi - \beta^2\xi = 0,$$

that is,

$$h^2b^2\xi + 2h(b\beta\xi - a\varepsilon^2) + \beta^2\xi - d\varepsilon^2 - \alpha\varepsilon^2 = 0.$$

The last equation is solvable with respect to h if $D \geq 0$, where

$$\begin{aligned} D &= 4(b\beta\xi - a\varepsilon^2)^2 - 4b^2\xi(\beta^2\xi - d\varepsilon^2 - \alpha\varepsilon^2) \\ &= 4b^2\beta^2\xi^2 - 8ab\beta\xi\varepsilon^2 + 4a^2\varepsilon^4 - 4b^2\xi^2\beta^2 + 4b^2\xi d\varepsilon^2 + 4b^2\xi\alpha\varepsilon^2 \\ &\overset{\text{by (3.63)}}{=} 4a^2\varepsilon^4 - 8ab\beta\xi\varepsilon^2 + 4b^2\xi\alpha\varepsilon^2 + 4b^2\xi\gamma\varepsilon^2 - 4b^2\varepsilon^4 \\ &= 4\varepsilon^4(a^2 - b^2) + 4b^2\xi\varepsilon^2(\alpha+\gamma) - 8ab\beta\xi\varepsilon^2 \geq 0. \end{aligned}$$

Consequently, (3.64) is solvable with respect to h if the inequality

$$\varepsilon^2(a^2 - b^2) + b^2\xi(\alpha+\gamma) \geq 2ab\beta\xi \tag{3.66}$$

holds. Then we could find the coefficient h from (3.64) and, subsequently, the coefficient c from (3.65).

If conditions (3.66), (3.33) and (3.61) are fulfilled, then all assumptions of Theorem 3.2.1 are fulfilled. Therefore, u_0 defined by the formula (3.62) is the desired optimal control function with the equation (3.59) taking the form

$$x'(t) = f(t, x_t, u_0) := \left(a - \frac{c}{\xi}(hc + \delta)\right) x(t) + \left(b - \frac{c\varepsilon}{\xi}\right) x(t - \tau). \tag{3.67}$$

Let $m = r = 1$ and $a = 2$, $b = -2$, $c = -4$, that is, let the system (3.3) be reduced to the scalar equation

$$x'(t) = f(t, x_t, u) := 2x(t) - 2x(t - \tau) - 4u, \tag{3.68}$$

where u is a control function. Set

$$\varepsilon = h = d = 1, \quad \xi = 2, \quad \alpha = 3, \quad \beta = \delta = 0, \quad \gamma = 3/2.$$

Conditions (3.33), (3.61), (3.63)–(3.65) for the coefficients are fulfilled and, by (3.62),

$$u_0 = 2x(t) - 0.5x(t - \tau).$$

Since $b - c\varepsilon/\xi = 0$, the equation (3.67) does not contain a delay and

$$x'(t) = f(t, x_t, 2x(t) - 0.5x(t - \tau)) := -6x(t). \tag{3.69}$$

Remark 3.3.7. The qualitative behaviour of the solutions of equation (3.68) without a control function, that is, the equation

$$x'(t) = 2(x(t) - x(t - \tau)), \tag{3.70}$$

is well-known and can be described using the results published, for example, in [21] and in a recent paper [20]. Assuming a solution of (3.70) in the exponential form $x = \exp(\lambda t)$ with a suitable constant λ, we arrive at the equation $\lambda = 2 - 2\exp(-\lambda\tau)$ which has a unique real root $\lambda = \lambda^* > 0$. For $t \to \infty$, every solution $x = x(t)$ of (3.70) has the following asymptotic representation

$$x(t) = K \exp(\lambda^* t) + \delta(t),$$

where K is a constant and $\delta(t)$ is a bounded solution of (3.70) (K and $\delta(t)$ depend on x). Note that the qualitative properties of the solutions of both equations - the controlled equation (3.69) and the equation without control (3.70), are diametrically opposite.

3.4 Application to linear equations and systems

In this part, we apply Theorem 3.2.1 to the linear equations and systems. Some auxiliary computations here are done by “WolframAlpha” software.

3.4.1 Equations with a single delay

Consider linear scalar equations with constant coefficients and a single delay

$$x'(t) = ax(t) + bx(t-\tau) + cu, \tag{3.71}$$

where a, $b \neq 0$, c are real constants, $\tau > 0$ is a delay and u is a control function. Together with the equation (3.71), we will consider a quality criterion (3.6) with

$$\omega(t, x_t, u) = \alpha x^2(t) + 2\beta x(t)x(t-\tau) + \gamma x^2(t-\tau) + \delta u^2, \tag{3.72}$$

i.e., (3.6) being a quadratic criterion

$$I = \int_{t_0}^{\infty} \Big(\alpha x^2(t) + 2\beta x(t)x(t-\tau) + \gamma x^2(t-\tau) + \delta u^2\Big)\mathrm{d}t, \tag{3.73}$$

with

$$\alpha > 0,\ \alpha\gamma - \beta^2 > 0,\ \delta > 0. \tag{3.74}$$

The equation (3.71) is formally the same as the equation (3.59). But the relevant quality criteria are different. Since $\varepsilon \neq 0$, the quality criterion (3.60) in Example 3.3.6 does not reduce to the quality criterion (3.72). In addition, in the latter case the coefficients of the Lyapunov-Krasovskii functional can be easily determined by simple formulas.

Theorem 3.4.1. *If, for the optimal control problem* (3.71), (3.73)*, a Lyapunov-Krasovskii functional is used in the form*

$$V(t, x_t) = hx^2(t) + d\int_{t-\tau}^{t} x^2(s)\mathrm{d}s,\ h > 0,\ d > 0,$$

with $h = -\beta/b$ ($\beta b < 0$), $d = \gamma$,

$$\delta(2ha + d + \alpha) - h^2c^2 = 0, \tag{3.75}$$

then the optimal stabilization control function u_0 equals

$$u_0 = -\frac{hc}{\delta}x(t). \tag{3.76}$$

Proof. We will employ Theorem 3.2.1. In accordance with the condition *iii)* of Theorem 3.2.1, we analyze the non-negativity of the expression B given by (3.11), i.e.,

$$\begin{aligned}
B\left(V,t,x_t,u\right) =&\frac{\mathrm{d}V(t,x_t)}{\mathrm{d}t}+\omega\left(t,x_t,u\right)\\
=&\frac{\mathrm{d}}{\mathrm{d}t}\left(hx^2(t)+d\int_{t-\tau}^{t}x^2(s)\mathrm{d}s\right)+\omega\left(t,x_t,u\right)\\
=&2hx(t)[ax(t)+bx(t-\tau)+cu]+d[x^2(t)-x^2(t-\tau)]\\
&+\alpha x^2(t)+2\beta x(t)x(t-\tau)+\gamma x^2(t-\tau)+\delta u^2.
\end{aligned}$$

Simplifying the last expression, we get

$$\begin{aligned}
B\left(V,t,x_t,u\right) =&(2ha+d+\alpha)x^2(t)+(\gamma-d)x^2(t-\tau)\\
&+(2hb+2\beta)x(t)x(t-\tau)+2hcx(t)u+\delta u^2.
\end{aligned}$$

For B to be non-negative, for any function u, the next inequalities should hold

$$2ha+d+\alpha \geq 0, \tag{3.77}$$

$$(2ha+d+\alpha)(\gamma-d)-(hb+\beta)^2 \geq 0, \tag{3.78}$$

$$\delta(2ha+d+\alpha)(\gamma-d)-h^2c^2(\gamma-d)-\delta(hb+\beta)^2 \geq 0. \tag{3.79}$$

For $u=u_0$, by *ii)*, we have

$$\begin{aligned}
B\left(V,t,x_t,u_0\right) =&(2ha+d+\alpha)x^2(t)+(\gamma-d)x^2(t-\tau)\\
&+(2hb+2\beta)x(t)x(t-\tau)+2hcx(t)u_0+\delta u_0^2=0.
\end{aligned}\tag{3.80}$$

Looking for an extremum of (3.80) with respect to u_0, we get

$$B'_{u_0}(V,t,x_t,u_0)=2hcx(t)+2\delta u_0=0,$$

i.e.,

$$u_0=-\frac{hc}{\delta}x(t),$$

which is the minimum of the function B because

$$B''_{u_0u_0}(V,t,x_t,u_0)=2\delta>0.$$

For (3.80) to hold, i.e.,

$$\begin{aligned}
&B\left(V,t,x_t,-\frac{hc}{\delta}x(t)\right)\\
&=\left(2ha+d+\alpha-\frac{h^2c^2}{\delta}\right)x^2(t)+(\gamma-d)x^2(t-\tau)+(2hb+2\beta)x(t)x(t-\tau)=0
\end{aligned}$$

we obtain

$$\begin{aligned}\delta(2ha + d + \alpha) - h^2c^2 &= 0,\\ d &= \gamma,\\ h &= -\frac{\beta}{b}.\end{aligned}$$

So the left-hand side of (3.77) takes the form

$$2ha + d + \alpha = \frac{h^2c^2}{\delta} \geq 0,$$

and the inequalities (3.78)-(3.79) are true because their left-hand sides are equal to zero.
Therefore,

$$u_0 = -\frac{hc}{\delta}x(t)$$

is the optimal stabilization control function. □

Remark 3.4.2. The equation (3.71) with $u = u_0$ given by (3.76) takes the form

$$x'(t) = \left(a - \frac{hc^2}{\delta}\right)x(t) + bx(t-\tau).$$

Example 3.4.3. Consider the equation (3.71) with $a = 1$, $b = -1$, $c = \sqrt{6}$, i.e.,

$$x'(t) = x(t) - x(t-\tau) + \sqrt{6}u \tag{3.81}$$

with the quadratic quality criterion (3.73) with $\alpha = 2 > 0$, $\beta = 1$, $\gamma = 2$, $\delta = 1 > 0$, $t_0 = 0$, i.e.,

$$I = \int_0^\infty \left(2x^2(t) + 2x(t)x(t-\tau) + 2x^2(t-\tau) + u^2\right)\mathrm{d}t.$$

Inequalities (3.74) are true (here $\alpha\gamma - \beta^2 = 3 > 0$), $h = 1$, $d = 2$ and (3.75) holds. By the formula (3.76), the optimal stabilization control function

$$u_0 = -\frac{hc}{\delta}x(t) = -\sqrt{6}x(t)$$

exists. The equation (3.81) with $u = u_0$ takes the form

$$x'(t) = -5x(t) - x(t-\tau).$$

3.4.2 Equations with multiple delays

Consider linear scalar equations with constant coefficients and delays

$$x'(t) = ax(t) + \sum_{i=1}^{n} b_i x(t-\tau_i) + cu, \;\; t \geq 0, \tag{3.82}$$

where a, b_i and c are real constants, $i = 1, \ldots, n$, $\tau_1 < \tau_2 < \cdots < \tau_n = \tau$ are delays and u is a control function.
Together with the equation (3.82), we will consider a quality criterion (3.6) with

$$\omega(t, x_t, u) = \sum_{i=0}^{n} \alpha_i x^2(t-\tau_i) + 2\sum_{i=1}^{n} \beta_i x(t)x(t-\tau_i) + \gamma u^2,$$

where $\tau_0 = 0$, i.e., (3.6) being a quadratic criterion

$$I = \int_{t_0}^{\infty} \left(\sum_{i=0}^{n} \alpha_i x^2(t-\tau_i) + 2\sum_{i=1}^{n} \beta_i x(t)x(t-\tau_i) + \gamma u^2 \right) \mathrm{d}t, \tag{3.83}$$

where α_0, α_i, β_i $(i = 1, \ldots, n)$ and $\gamma > 0$ are constants and the matrix

$$\begin{pmatrix} \alpha_0 & \beta_1 & \beta_2 & \ldots & \beta_n \\ \beta_1 & \alpha_1 & 0 & \ldots & 0 \\ \beta_2 & 0 & \alpha_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_n & 0 & 0 & \ldots & \alpha_n \end{pmatrix} \tag{3.84}$$

is positive-definite.

Theorem 3.4.4. *Let*

$$-\beta_i / b_i = h > 0, \;\; i = 1, \ldots, n. \tag{3.85}$$

If for the optimal control problem (3.82), (3.83) *a Lyapunov-Krasovskii functional is used in the form*

$$V(t, x_t) = hx^2(t) + \sum_{i=1}^{n} d_i \int_{t-\tau_i}^{t} x^2(s)\mathrm{d}s, \;\; h > 0, \;\; d_i > 0,$$

with

$$d_i = \alpha_i \tag{3.86}$$

and if

$$\gamma \left(2ha + \sum_{i=1}^{n} d_i + \alpha_0 \right) - h^2 c^2 = 0, \tag{3.87}$$

then the optimal stabilization control function u_0 equals

$$u_0 = -\frac{hc}{\gamma} x(t). \tag{3.88}$$

Proof. We will employ Theorem 3.2.1. In accordance with the conditions *ii*), *iii*) of Theorem 3.2.1, we analyze the expression B given by (3.11), i.e.,

$$
\begin{aligned}
B\left(V, t, x_t, u\right) =& \frac{\mathrm{d}V(t, x_t)}{\mathrm{d}t} + \omega\left(t, x_t, u\right) \\
=& \frac{\mathrm{d}}{\mathrm{d}t}\left(hx^2(t) + \sum_{i=1}^{n} d_i \int_{t-\tau_i}^{t} x^2(s)\mathrm{d}s\right) + \omega\left(t, x_t, u\right) \\
=& 2hx(t)\left[ax(t) + \sum_{i=1}^{n} b_i x(t-\tau_i) + cu\right] + \sum_{i=1}^{n} d_i \left(x^2(t) - x^2(t-\tau_i)\right) \\
& + \alpha_0 x^2(t) + \sum_{i=1}^{n} \alpha_i x^2(t-\tau_i) + 2x(t)\sum_{i=1}^{n} \beta_i x(t-\tau_i) + \gamma u^2.
\end{aligned}
$$

Simplifying the last expression, we get

$$
\begin{aligned}
B\left(V, t, x_t, u\right) = & \left(2ha + \sum_{i=1}^{n} d_i + \alpha_0\right) x^2(t) + \sum_{i=1}^{n}(\alpha_i - d_i)x^2(t-\tau_i) \\
& + 2x(t)\sum_{i=1}^{n}(hb_i + \beta_i)x(t-\tau_i) + 2hcx(t)u + \gamma u^2.
\end{aligned} \tag{3.89}
$$

Looking for an extremum of (3.89), we get

$$
B'_u(V, t, x_t, u) = 2hcx(t) + 2\gamma u = 0,
$$

i.e.,

$$
u = -\frac{hc}{\gamma}x(t), \tag{3.90}
$$

which is the minimum of the function B because

$$
B''_{uu}(V, t, x_t, u) = 2\gamma > 0.
$$

For $B(V, t, x_t, u) = 0$ to hold, use (3.89)

$$
\begin{aligned}
& B\left(V, t, x_t, -\frac{hc}{\delta}x(t)\right) \\
& \qquad = \left(2ha + \sum_{i=1}^{n} d_i + \alpha_0 - \frac{h^2c^2}{\gamma}\right) x^2(t) + \sum_{i=1}^{n}(\alpha_i - d_i)x^2(t-\tau_i) \\
& \qquad\quad + 2x(t)\sum_{i=1}^{n}(hb_i + \beta_i)x(t-\tau_i) = 0
\end{aligned}
$$

we obtain conditions

$$
\gamma\left(2ha + \sum_{i=1}^{n} d_i + \alpha_0\right) - h^2c^2 = 0, \tag{3.91}
$$

$$
d_i = \alpha_i, \ i = 1, \dots, n \tag{3.92}
$$

$$
h = -\frac{\beta_i}{b_i}, \ i = 1, \dots, n. \tag{3.93}
$$

Since (3.85)–(3.87) hold, so do (3.91)–(3.93).
For the non-negativity of B, by iii), and using (3.92)-(3.93) we obtain:

$$B\left(V, t, x_t, u\right) = \left(2ha + \sum_{i=1}^{n} d_i + \alpha_0\right) x^2(t) + 2hcx(t)u + \gamma u^2 \geq 0.$$

So we need

$$2ha + \sum_{i=1}^{n} d_i + \alpha_0 > 0,$$

which, by (3.91), takes the form

$$2ha + \sum_{i=1}^{n} d_i + \alpha_0 = \frac{h^2c^2}{\gamma} > 0.$$

We conclude that (3.90) gives an optimal stabilization control function, that is,

$$u_0 = -\frac{hc}{\gamma} x(t).$$

□

Remark 3.4.5. The equation (3.82) with $u = u_0$ given by (3.88) takes the form

$$x'(t) = \left(a - \frac{hc^2}{\gamma}\right) x(t) + \sum_{i=1}^{n} b_i x(t - \tau_i).$$

Example 3.4.6. Let $n = 2$. Consider the equation (3.82) with $a = -3$, $b_1 = -1$, $b_2 = -1$, $c = 1$, i.e.,

$$x'(t) = -3x(t) - x(t - \tau_1) - x(t - \tau_2) + u \tag{3.94}$$

with the quadratic quality criterion (3.83) with $\alpha_0 = 3$, $\alpha_1 = \alpha_2 = 2$, $\beta_1 = \beta_2 = \gamma = 1$ and $t_0 = 0$, i.e.,

$$I = \int_0^\infty \left(2x^2(t) + 2x(t)x(t-\tau) + 2x^2(t-\tau) + 2x^2(t-\delta) + 2x(t)x(t-\delta) + u^2\right) \mathrm{d}t,$$

where the matrix (3.84), that is

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

is positive-definite. Since $-\beta_1/b_1 = -\beta_2/b_2 = h = 1 > 0$ and

$$\gamma\left(2ha + \alpha_0 + \alpha_1 + \alpha_2\right) - h^2c^2 = -6 + 7 - 1 = 0,$$

all the assumptions of Theorem 3.4.4 are fulfilled. By the formula (3.88), the optimal stabilization control function

$$u_0 = -\frac{hc}{\gamma} x(t) = -x(t)$$

exists and the equation (3.94) with $u = u_0$ takes the form

$$x'(t) = -4x(t) - x(t - \tau_1) - x(t - \tau_2).$$

3.4.3 Systems with a single delay and a scalar control function

Consider linear systems with constant coefficients and a single constant delay

$$x'(t) = A_0 x(t) + A_1 x(t-\tau) + bu, \tag{3.95}$$

where A_0, A_1 are $m \times m$ constant matrices, $b \in \mathbb{R}^m$, $u \in \mathbb{R}$, and a quality criterion (3.6) with

$$\begin{aligned}\omega(t, x_t, u) = x^T(t)C_{11}x(t) + x^T(t)C_{12}x(t-\tau)&\\ + x^T(t-\tau)C_{21}x(t) + x^T(t-\tau)C_{22}x(t-\tau) + du^2&,\end{aligned}$$

where $m \times m$ matrices C_{11}, C_{22} and an $2m \times 2m$ matrix

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \tag{3.96}$$

are positive-definite and symmetric, $C_{21} = C_{12}^T$ and $d > 0$, i.e., (3.6) is a quadratic criterion

$$\begin{aligned}I = \int_{t_0}^{\infty} \Big(x^T(t)C_{11}x(t) + x^T(t)C_{12}x(t-\tau)&\\ + x^T(t-\tau)C_{21}x(t) + x^T(t-\tau)C_{22}x(t-\tau) + du^2 \Big)\,\mathrm{d}t&.\end{aligned} \tag{3.97}$$

We will employ a Lyapunov-Krasovskii functional

$$V(t, x_t) = x^T(t)Hx(t) + \int_{t-\tau}^{t} x^T(s)Gx(s)\mathrm{d}s, \tag{3.98}$$

where H and G are $m \times m$, constant, positive-definite and symmetric matrices.

Theorem 3.4.7. *Assume that there exists a positive-definite symmetric $m \times m$ matrix H satisfying a matrix equation*

$$A_0^T H + HA_0 + C_{11} + C_{22} - \frac{1}{d}Hbb^T H = \Theta_{m\times m}. \tag{3.99}$$

If, moreover,

$$HA_1 + C_{12} = \Theta_{m\times m}, \tag{3.100}$$

the optimal stabilization control function $u = u_0$ of the problem (3.95), (3.97) *exists and*

$$u_0 = -\frac{1}{d}b^T Hx(t). \tag{3.101}$$

Proof. We will use Theorem 3.2.1. In accordance with the conditions *ii*), *iii*) of Theorem 3.2.1, we analyse the expression B given by (3.11), i.e.,

$$\begin{aligned}
B\left(V,t,x_t,u\right) =&\frac{\mathrm{d}V(t,x_t)}{\mathrm{d}t}+\omega\left(t,x_t,u\right)\\
=&\frac{\mathrm{d}}{\mathrm{d}t}\left(x^T(t)Hx(t)+\int_{t-\tau}^{t}x^T(s)Gx(s)\mathrm{d}s\right)+\omega\left(t,x_t,u\right)\\
=&[A_0x(t)+A_1x(t-\tau)+bu]^THx(t)\\
&+x^T(t)H[A_0x(t)+A_1x(t-\tau)+bu]+x^T(t)Gx(t)\\
&-x^T(t-\tau)Gx(t-\tau)+x^T(t)C_{11}x(t)+x^T(t)C_{12}x(t-\tau)\\
&+x^T(t-\tau)C_{21}x(t)+x^T(t-\tau)C_{22}x(t-\tau)+du^2.
\end{aligned}$$

Simplifying the last expression, we get

$$\begin{aligned}
B\left(V,t,x_t,u\right) =&x^T(t)[A_0^TH+HA_0+G+C_{11}]x(t)+x^T(t-\tau)[A_1^TH+C_{21}]x(t)\\
&+x^T(t)[HA_1+C_{12}]x(t-\tau)+x^T(t-\tau)[C_{22}-G]x(t-\tau)\\
&+2x^T(t)Hbu+du^2.
\end{aligned}\tag{3.102}$$

Looking for an extremum of (3.102) with regard to u, we get

$$B_u'(V,t,x_t,u)=2x^T(t)Hb+2du,$$

i.e.,

$$u=-\frac{1}{d}x^T(t)Hb=-\frac{1}{d}b^THx(t),\tag{3.103}$$

which is the minimum of the function B because

$$B_{uu}''(V,t,x_t,u)=2d>0.$$

For (3.12) to hold, i.e., for

$$\begin{aligned}
B\left(V,t,x_t,u_0\right) =&x^T(t)[A_0^TH+HA_0+G+C_{11}]x(t)\\
&+x^T(t-\tau)[A_1^TH+C_{21}]x(t)+x^T(t)[HA_1+C_{12}]x(t-\tau)\\
&+x^T(t-\tau)[C_{22}-G]x(t-\tau)-\frac{1}{d}x^T(t)Hbb^THx(t)\\
=&x^T(t)\left[A_0^TH+HA_0+G+C_{11}-\frac{1}{d}Hbb^TH\right]x(t)\\
&+x^T(t-\tau)[A_1^TH+C_{21}]x(t)+x^T(t)[HA_1+C_{12}]x(t-\tau)\\
&+x^T(t-\tau)[C_{22}-G]x(t-\tau)\equiv 0
\end{aligned}$$

we obtain

$$A_0^TH+HA_0+G+C_{11}-\frac{1}{d}Hbb^TH=\Theta_{m\times m},$$

$$
\begin{aligned}
HA_1 + C_{12} =& \Theta_{m\times m},\\
C_{22} =& G.
\end{aligned}
$$

If the above conditions are fulfilled, the control function (3.103) defines an optimal stabilization control function, the system (3.95) is asymptotically stable, and the quality criterion (3.97) takes a minimum value. □

Remark 3.4.8. The equation (3.95) with $u = u_0$ given by (3.101) takes the form

$$
x'(t) = \left(A_0 - \frac{1}{d} b b^T H\right) x(t) + A_1 x(t-\tau).
$$

Example 3.4.9. Consider the system (3.95) with $m = r = 2$ and

$$
A_0 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1 & -0.1 \\ -0.5 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix},
$$

that is

$$
\begin{aligned}
x_1'(t) =& -2x_1(t) + x_2(t) - x_1(t-\tau) - 0.1x_2(t-\tau) + u,\\
x_2'(t) =& \quad x_1(t) - 2x_2(t) - 0.5x_1(t-\tau) - x_2(t-\tau) + u
\end{aligned}
\tag{3.104}
$$

with the quadratic quality criterion (3.97) with $t_0 = 0$ and

$$
C_{11} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \quad C_{12} = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}, \quad C_{21} = \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix}, \quad C_{22} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \quad d = 1,
$$

i.e.,

$$
\begin{aligned}
I&\\
=& \int_0^\infty \left(\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}^T \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}^T \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \begin{pmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{pmatrix} \right.\\
&\left. + \begin{pmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{pmatrix}^T \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{pmatrix}^T \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{pmatrix} + u^2 \right) \mathrm{d}t\\
=& \int_0^\infty \Big(3x_1^2(t) + 3x_2^2(t) + 2c_1 x_1(t) x_1(t-\tau_1) + 2c_3 x_1(t-\tau_1) x_2(t)\\
&+ 2c_2 x_1(t) x_2(t-\tau_1) + 2c_4 x_2(t) x_2(t-\tau_1) + 3x_1^2(t-\tau_1) + 3x_2^2(t-\tau_1) + u^2 \Big) \mathrm{d}t
\end{aligned}
$$

By the formula (3.101) we obtain the optimal stabilization control function in the form

$$
u_0 = -\frac{1}{d} b^T H x(t) = -\begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -(h_1 + h_2) x_1 - (h_2 + h_3) x_2. \tag{3.105}
$$

We need to find a suitable matrix H such that (3.99) and (3.100) will hold.

$$A_0^T H + HA_0 + C_{11} + C_{22} - \frac{1}{d} Hbb^T H$$
$$= \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}^T \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix} + \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$
$$- \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix}$$
$$= \begin{pmatrix} -4h_1 + 2h_2 + 6 - (h_1+h_2)^2 & h_1 - 4h_2 + h_3 - (h_1+h_2)(h_2+h_3) \\ h_1 - 4h_2 + h_3 - (h_1+h_2)(h_2+h_3) & 2h_2 - 4h_3 + 6 - (h_2+h_3)^2 \end{pmatrix}$$
$$= \Theta_{2\times 2}.$$

It means that

$$\begin{cases} -4h_1 + 2h_2 + 6 - (h_1+h_2)^2 = 0, & (3.106) \\ h_1 - 4h_2 + h_3 - (h_1+h_2)(h_2+h_3) = 0, & (3.107) \\ 2h_2 - 4h_3 + 6 - (h_2+h_3)^2 = 0. & (3.108) \end{cases}$$

To solve the above system we can, for example, add the second equation multiplied by 2 to the sum of the first and the third equations ((3.106)+2(3.107)+(3.108)). We obtain

$$-2h_1 - 4h_2 - 2h_3 + 12 - [(h_1+h_2) + (h_2+h_3)]^2$$
$$= -2[h_1 + 2h_2 + h_3] + 12 - [h_1 + 2h_2 + h_3]^2 = 0.$$

If put

$$h_1 + 2h_2 + h_3 = K, \tag{3.109}$$

then we have

$$K^2 + 2K - 12 = 0$$

and $K = -1 \pm \sqrt{13}$.

After subtracting the first equation of the system from the third one, i.e., ((3.108)–(3.106)), we obtain

$$4h_1 - 4h_3 + (h_1+h_2)^2 - (h_2+h_3)^2$$
$$= 4(h_1 - h_3) + (h_1 + 2h_2 + h_3)(h_1 - h_3) = (h_1 - h_3)(4 + K) = 0$$

and, therefore,

$$h_1 = h_3. \tag{3.110}$$

Using (3.110) to (3.109), we find

$$h_1 + h_2 = \frac{K}{2}. \tag{3.111}$$

For the second equation of the system, i.e., for (3.107), we obtain

$$2h_1 - 4h_2 - (h_1 + h_2)^2 = 0 \Rightarrow h_1 - 2h_2 = \frac{K^2}{8}. \tag{3.112}$$

From (3.111) and (3.112) we find that

$$h_1 = h_3 = \frac{K}{3} + \frac{K^2}{24},$$

$$h_2 = \frac{K}{6} - \frac{K^2}{24}.$$

For $K = -1 - \sqrt{13}$, the matrix H is not positive-definite, so

$$H = \begin{pmatrix} \dfrac{1+\sqrt{13}}{4} & \dfrac{\sqrt{13}-3}{4} \\ \dfrac{\sqrt{13}-3}{4} & \dfrac{1+\sqrt{13}}{4} \end{pmatrix}.$$

The condition (3.100) should hold as well so that

$$C_{12} = C_{21}^T = -HA_1 = \begin{pmatrix} 1.22708 & 0.266527 \\ 0.727082 & 1.16653 \end{pmatrix},$$

which is sufficient for (3.96) to be a positive-definite matrix. By (3.105) the optimal stabilization control function will be

$$u_0 = \frac{1-\sqrt{13}}{2}(x_1(t) + x_2(t)),$$

with the system (3.104) taking the form (the coefficients of non-delayed terms are computed approximately)

$$\begin{aligned} x_1'(t) &= -\,3.3x_1(t) - 0.3x_2(t) - x_1(t-\tau) - 0.1x_2(t-\tau), \\ x_2'(t) &= -\,0.3x_1(t) - 3.3x_2(t) - 0.5x_1(t-\tau) - x_2(t-\tau). \end{aligned}$$

3.4.4 Systems with a single delay and a control vector-function

Consider linear systems with constant coefficients and a single constant delay

$$x'(t) = A_0x(t) + A_1x(t-\tau) + Pu, \tag{3.113}$$

where A_0, A_1 are $m \times m$ constant matrices, $P \in \mathbb{R}^{m\times r}$, $u \in \mathbb{R}^r$, and a quality criterion (3.6)

$$I = \int_{t_0}^{\infty} (x^T(t)C_{11}x(t) + x^T(t)C_{12}x(t-\tau)$$

$$+ x^T(t-\tau)C_{21}x(t) + x^T(t-\tau)C_{22}x(t-\tau) + u^T Du)\mathrm{d}t, \tag{3.114}$$

where $m \times m$ matrices C_{11}, C_{22} and an $2m \times 2m$ matrix (3.96), i.e.,

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

are positive-definite and symmetric, $C_{21} = C_{12}^T$ and D is a diagonal matrix, $D = \mathrm{diag}\{d_j\}$, $d_j > 0$, $j = 1, \dots, r$.
We will use a Lyapunov-Krasovskii functional (3.98), that is

$$V(t, x_t) = x^T(t)Hx(t) + \int_{t-\tau}^{t} x^T(s)Gx(s)\mathrm{d}s,$$

where H and G are $m \times m$ constant, positive-definite and symmetric matrices.

Theorem 3.4.10. *Assume that there exists a positive-definite symmetric matrix H satisfying the matrix equation*

$$A_0^T H + HA_0 + C_{11} + C_{22} - HPD^{-1}P^T H = \Theta_{m\times m}. \tag{3.115}$$

If, moreover,

$$HA_1 + C_{12} = \Theta_{m\times m}, \tag{3.116}$$

the optimal stabilization control function $u = u_0$ of the problem (3.113), (3.114) *exists and*

$$u_0 = -D^{-1}P^T Hx(t). \tag{3.117}$$

Proof. By the conditions *ii*), *iii*) of Theorem 3.2.1 we analyse the expression B given by (3.11), i.e.,

$$\begin{aligned}
B(V, t, x_t, u) =& \frac{\mathrm{d}V(t, x_t)}{\mathrm{d}t} + \omega(t, x_t, u) \\
=& \frac{\mathrm{d}}{\mathrm{d}t}\left(x^T(t)Hx(t) + \int_{t-\tau}^{t} x^T(s)Gx(s)\mathrm{d}s\right) + \omega(t, x_t, u) \\
=& [A_0x(t) + A_1x(t-\tau) + Pu]^T Hx(t) \\
& + x^T(t)H[A_0x(t) + A_1x(t-\tau) + Pu(t)] + x^T(t)Gx(t) \\
& - x^T(t-\tau)Gx(t-\tau) + x^T(t)C_{11}x(t) + x^T(t)C_{12}x(t-\tau) \\
& + x^T(t-\tau)C_{21}x(t) + x^T(t-\tau)C_{22}x(t-\tau) + u^T Du.
\end{aligned}$$

Simplifying the last expression, we get

$$\begin{aligned}
B(V, t, x_t, u) =& x^T(t)[A_0^T H + HA_0 + G + C_{11}]x(t) + x^T(t-\tau)[A_1^T H + C_{21}]x(t) \\
& + x^T(t)[HA_1 + C_{12}]x(t-\tau) + x^T(t-\tau)[C_{22} - G]x(t-\tau)
\end{aligned}$$

$$+ 2x^T(t)HPu + u^T Du. \tag{3.118}$$

Looking for an extremum of (3.118), with respect to u, we get

$$B'_u(V, t, x_t, u) = 2P^T Hx(t) + 2Du = 0,$$

i.e.,

$$u = -D^{-1}P^T Hx(t), \tag{3.119}$$

which is the minimum of the function B because the matrix $B''_{uu} = 2D > 0$. For (3.12) to hold, i.e., for

$$\begin{aligned} B\left(V, t, x_t, u_0\right) =& x^T(t)[A_0^T H + HA_0 + G + C_{11} - HPD^{-1}P^T H]x(t) \\ &+ x^T(t-\tau)[A_1^T H + C_{21}]x(t) + x^T(t)[HA_1 + C_{12}]x(t-\tau) \\ &+ x^T(t-\tau)[C_{22} - G]x(t-\tau) \equiv 0 \end{aligned}$$

it is necessary that

$$\begin{gathered} A_0^T H + HA_0 + G + C_{11} - HPD^{-1}P^T H = \Theta_{m\times m}, \\ HA_1 + C_{12} = \Theta_{m\times m}, \\ C_{22} = G. \end{gathered}$$

If the above conditions are fulfilled, the control function (3.119) is the desired optimal stabilization control function (3.117), the system (3.113) is asymptotically stable and the quality criterion (3.114) takes a minimum value. □

Remark 3.4.11. The equation (3.113) with $u = u_0$ given by (3.117) takes the form

$$x'(t) = \left(A_0 - PD^{-1}P^T H\right) x(t) + A_1 x(t-\tau).$$

Example 3.4.12. Consider the system (3.113) with the quality criterion (3.114). Let $t_0 = 0$ and the matrices have the form

$$A_0 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \; A_1 = \begin{pmatrix} -1 & -0.1 \\ -0.5 & -1 \end{pmatrix}, \; P = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix},$$

where ε is an arbitrary constant, $\varepsilon \neq \pm 1$, i.e.,

$$\begin{aligned} x_1'(t) &= -2x_1(t) + x_2(t) - x_1(t-\tau) - 0.1x_2(t-\tau) + u_1 + \varepsilon u_2, \\ x_2'(t) &= x_1(t) - 2x_2(t) - 0.5x_1(t-\tau) - x_2(t-\tau) + \varepsilon u_1 + u_2 \end{aligned} \tag{3.120}$$

and

$$C_{11} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \; C_{12} = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}, \; C_{21} = \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix}, \; C_{22} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \; D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

i.e.,

$$
\begin{aligned}
I = \int_0^\infty \Bigg(&\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}^T \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}^T \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \begin{pmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{pmatrix} \\
&+ \begin{pmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{pmatrix}^T \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{pmatrix}^T \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{pmatrix} \\
&+ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Bigg) \mathrm{d}t \\
= \int_0^\infty &\Big(3x_1^2(t) + 3x_2^2(t) + 2c_1 x_1(t) x_1(t-\tau_1) + 2c_3 x_1(t-\tau_1) x_2(t) \\
&+ 2c_2 x_1(t) x_2(t-\tau_1) + 2c_4 x_2(t) x_2(t-\tau_1) + 3x_1^2(t-\tau_1) + 3x_2^2(t-\tau_1) + u_1^2 + u_2^2\Big) \mathrm{d}t.
\end{aligned}
$$

By (3.117) the optimal control function will be in the form

$$
u_0 = -D^{-1}P^T H x(t) = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}^T \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},
$$

that is,

$$
\begin{aligned}
u_1^0 &= -(h_1 + h_2)x_1 - (h_2 + h_3)x_2, \\
u_2^0 &= -(h_1 + h_2)x_1 - (h_2 + h_3)x_2.
\end{aligned} \tag{3.121}
$$

We need to find a suitable matrix H such that (3.115), (3.116) will hold. In our case (3.115) equals

$$
\begin{gathered}
A_0^T H + H A_0 + C_{11} + C_{22} - HPD^{-1}P^T H \\
= \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}^T \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix} + \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \\
- \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix} \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}^T \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix} = \Theta_{2\times 2}
\end{gathered}
$$

which means that

$$
\begin{cases}
-4h_1 + 2h_2 + 6 - (h_1 + \varepsilon h_2)^2 - (\varepsilon h_1 + h_2)^2 = 0, & (3.122) \\
h_1 - 4h_2 + h_3 - (h_1 + \varepsilon h_2)(h_2 + \varepsilon h_3) - (\varepsilon h_1 + h_2)(\varepsilon h_2 + h_3) = 0, & (3.123) \\
2h_2 - 4h_3 + 6 - (h_2 + \varepsilon h_3)^2 - (\varepsilon h_2 + h_3)^2 = 0. & (3.124)
\end{cases}
$$

To solve the above system we can, for example, subtract the first equation from the third one, i.e., ((3.124)–(3.122)). We obtain

$$
\begin{aligned}
&4h_1 - 4h_3 + (h_1 + \varepsilon h_2)^2 - (h_2 + \varepsilon h_3)^2 + (\varepsilon h_1 + h_2)^2 - (\varepsilon h_2 + h_3)^2 \\
&= 4(h_1 - h_3) + (h_1 + \varepsilon h_2 + h_2 + \varepsilon h_3)(h_1 + \varepsilon h_2 - h_2 - \varepsilon h_3)
\end{aligned}
$$

$$
\begin{aligned}
&+ (\varepsilon h_1 + h_2 + \varepsilon h_2 + h_3)(\varepsilon h_1 + h_2 - \varepsilon h_2 - h_3)\\
=&4(h_1 - h_3) + h_2(1+\varepsilon)(h_1 - \varepsilon h_3 + \varepsilon h_1 - h_3)\\
&+ (h_1 + \varepsilon h_3)(h_1 + \varepsilon h_2 - h_2 - \varepsilon h_3) + (\varepsilon h_1 + h_3)(\varepsilon h_1 + h_2 - \varepsilon h_2 - h_3)\\
=&4(h_1 - h_3) + h_2(1+\varepsilon)^2(h_1 - h_3)\\
&+ h_1^2(1+\varepsilon^2) + h_1 h_2(2\varepsilon - 1 - \varepsilon^2) + h_2 h_3(\varepsilon^2 - 2\varepsilon + 1) + h_3^2(-\varepsilon^2 - 1)\\
=&(h_1 - h_3)(4 + h_2(1+\varepsilon^2)) + (1+\varepsilon^2)(h_1^2 - h_3^2) - h_2(\varepsilon - 1)^2(h_1 - h_3)\\
=&(h_1 - h_3)(4 + h_2(1+\varepsilon^2) + (h_1 + h_3)(1+\varepsilon^2) - h_2(\varepsilon - 1)^2)\\
=&(h_1 - h_3)(4 + 2h_2\varepsilon + (h_1 + h_3)(1+\varepsilon^2)) = 0.
\end{aligned}
$$

This implies

$$h_1 = h_3 \tag{3.125}$$

since

$$4 + 2h_2\varepsilon + (h_1 + h_3)(1+\varepsilon^2) > 0$$

(this inequality holds since the matrix H is positive-definite and $h_1 > |h_2|$).
We add the second equation multiplied by 2 to the sum of the first and the third equations ((3.122)+2(3.123)+(3.124)) to obtain

$$-2h_1 - 4h_2 - 2h_3 + 12 - (h_1 + \varepsilon h_2 + h_2 + \varepsilon h_3)^2 - (\varepsilon h_1 + h_2 + \varepsilon h_2 + h_3)^2 = 0$$

and, using (3.125), we get

$$-4(h_1 + h_2) + 12 - 2(h_1 + h_2)^2(1+\varepsilon)^2 = 0.$$

If we put

$$h_1 + h_2 = K > 0, \tag{3.126}$$

then we have

$$K^2(1+\varepsilon)^2 + 2K - 6 = 0 \tag{3.127}$$

and

$$K = \frac{-1 + \sqrt{1 + 6(1+\varepsilon)^2}}{(1+\varepsilon)^2}. \tag{3.128}$$

Using (3.125) and rewriting (3.126), i.e.,

$$h_1 = K - h_2 \Rightarrow K > h_2, \tag{3.129}$$

for (3.123) we obtain

$$
\begin{aligned}
&h_1 - 2h_2 - (h_1 + \varepsilon h_2)(\varepsilon h_1 + h_2)\\
&\quad = K - 3h_2 - (K - h_2 + \varepsilon h_2)(\varepsilon K - \varepsilon h_2 + h_2) = 0.
\end{aligned}
$$

After simplification, we obtain the following equation

$$h_2^2(\varepsilon-1)^2 + h_2(-K(\varepsilon-1)^2 - 3) + K - K^2\varepsilon = 0,$$

where

$$h_2 = \frac{K(\varepsilon-1)^2 + 3 \pm \sqrt{(K(\varepsilon-1)^2+3)^2 - 4(\varepsilon-1)^2(K-K^2\varepsilon)}}{2(\varepsilon-1)^2}$$

and

$$\begin{aligned}
&(K(\varepsilon-1)^2+3)^2 - 4(\varepsilon-1)^2(K-K^2\varepsilon) \\
&\quad = K^2(\varepsilon-1)^4 + 6K(\varepsilon-1)^2 + 9 - 4(\varepsilon-1)^2(K-K^2\varepsilon) \\
&\quad = K^2(\varepsilon-1)^2((\varepsilon-1)^2+4\varepsilon) + 2K(\varepsilon-1)^2 + 9 \\
&\quad = K^2(\varepsilon-1)^2(\varepsilon+1)^2 + 2K(\varepsilon-1)^2 + 9 \\
&\quad \overset{\text{by (3.127)}}{=} (\varepsilon-1)^2(-2K+6) + 2K(\varepsilon-1)^2 + 9 = 6(\varepsilon-1)^2 + 9.
\end{aligned}$$

So

$$h_2 = \frac{K(\varepsilon-1)^2 + 3 \pm \sqrt{6(\varepsilon-1)^2+9}}{2(\varepsilon-1)^2}. \tag{3.130}$$

For, say, $\varepsilon = 0.5$, we obtain approximate values from (3.128), (3.130), (3.129), (3.125)

$$K \doteq 1.24795, \quad h_2 \doteq 0.143234, \quad h_1 = h_3 \doteq 1.104716$$

(another solution for $h_2 \doteq 13.1047$ does not satisfy (3.129)). Moreover, (3.116) should hold as well so that

$$C_{12} = C_{21}^T = -HA_1 = \begin{pmatrix} 1.17633 & 0.253706 \\ 0.695592 & 1.11904 \end{pmatrix},$$

which is sufficient for (3.96) to be a positive-definite matrix.
By (3.121), the optimal stabilization control function will be (the coefficients are computed approximately)

$$u_1^0 = -1.176333x_1(t) - 0.695592x_2(t),$$

$$u_2^0 = -0.695592x_1(t) - 1.176333x_2(t),$$

with the system (3.120) taking the form

$$\begin{aligned}
x_1'(t) &= -\,3.52413\ \ x_1(t) - 0.283759x_2(t) - \quad\ x_1(t-\tau) - 0.1x_2(t-\tau), \\
x_2'(t) &= -\,0.283759x_1(t) - 3.52413\ \ x_2(t) - 0.5x_1(t-\tau) - \quad\ x_2(t-\tau).
\end{aligned}$$

3.4.5 Systems with multiple delays and a scalar control function

In this part, we consider systems of linear differential equations with delays

$$x'(t) = \sum_{i=0}^{n} A_i x(t-\tau_i) + cu, \ \ t \geq t_0, \tag{3.131}$$

where A_i, $i = 0, \dots, n$ are $m \times m$ real matrices, $c \in \mathbb{R}^m$, $0 = \tau_0 < \tau_1 < \cdots < \tau_n$, $x \colon [t_0 - \tau, \infty) \to \mathbb{R}^m$, $t_0 \in \mathbb{R}$ and $u \in \mathbb{R}$ is a control function. Set $\tau := \tau_n$. A minimization problem (3.6) with

$$\begin{aligned} \omega\left(t, x_t, u\right) := \sum_{i=0}^{n} x^T(t-\tau_i) C_{ii} x(t-\tau_i) + \sum_{i=1}^{n} x^T(t) C_{0i} x(t-\tau_i) \\ + \sum_{i=1}^{n} x^T(t-\tau_i) C_{i0} x(t) + du^2 \end{aligned} \tag{3.132}$$

will be solved for the system (3.131), where constant symmetric $m \times m$ matrices C_{ii} and an auxiliary matrix

$$C = \begin{pmatrix} C_{00} & C_{01} & \dots & C_{0n} \\ C_{10} & C_{11} & \dots & C_{1n} \\ \vdots & \vdots & \ddots & \\ C_{n0} & C_{n1} & \dots & C_{nn} \end{pmatrix}, \tag{3.133}$$

(with $C_{ij} = C_{ji} = \Theta_{m \times m}$, $i > j \geq 1$, $i, j = 1, \dots, n$) are positive-definite, C_{0i} and C_{i0}, $C_{0i} = C_{i0}^T$ are $m \times m$ constant matrices, $d > 0$. We will employ a Lyapunov-Krasovskii functional

$$V(t, x_t) = x^T(t) H x(t) + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} x^T(s) G_i x(s) \mathrm{d}s, \tag{3.134}$$

where $m \times m$ matrices H and G_i, $i = 1, \dots, n$ are constant, positive-definite and symmetric.

Theorem 3.4.13. *Assume that the matrix C is positive-definite and there exists a positive-definite symmetric matrix H satisfying the matrix equation*

$$A_0^T H + H A_0 + \sum_{i=0}^{n} C_{ii} - \frac{1}{d} H c c^T H = \Theta_{m \times m}. \tag{3.135}$$

If, moreover,

$$A_i^T H + C_{i0} = \Theta_{m \times m}, \ \ i = 1, \dots, n \tag{3.136}$$

then the optimal stabilization control function of the problem (3.131), (3.132) *exists and equals*

$$u_0 = -\frac{1}{d} c^T H x(t). \tag{3.137}$$

Proof. By Theorem 3.2.1 we will analyze the expression B given by (3.11). We get

$$\begin{aligned}
B(V,&t,x_t,u) = \frac{\mathrm{d}V(t,x_t)}{\mathrm{d}t} + \omega\left(t,x_t,u\right) \\
&= \left[\sum_{i=0}^{n} A_i x(t-\tau_i) + cu\right]^T Hx(t) + x^T(t)H\left[\sum_{i=0}^{n} A_i x(t-\tau_i) + cu\right] \\
&\quad + \sum_{i=1}^{n}\left[x^T(t)G_i x(t) - x^T(t-\tau_i)G_i x(t-\tau_i)\right] + \sum_{i=0}^{n} x^T(t-\tau_i)C_{ii}x(t-\tau_i) \\
&\quad + \sum_{i=1}^{n} x^T(t)C_{0i}x(t-\tau_i) + \sum_{i=1}^{n} x^T(t-\tau_i)C_{i0}x(t) + du^2 \\
&= \left[A_0x(t) + \sum_{i=1}^{n} A_i x(t-\tau_i) + cu\right]^T Hx(t) + x^T(t)H\left[A_0x(t) + \sum_{i=1}^{n} A_i x(t-\tau_i) + cu\right] \\
&\quad + \sum_{i=1}^{n}\left[x^T(t)G_i x(t) - x^T(t-\tau_i)G_i x(t-\tau_i)\right] + x^T(t)C_{00}x(t) \\
&\quad + \sum_{i=1}^{n} x^T(t-\tau_i)C_{ii}x(t-\tau_i) + \sum_{i=1}^{n} x^T(t)C_{0i}x(t-\tau_i) + \sum_{i=1}^{n} x^T(t-\tau_i)C_{i0}x(t) + du^2.
\end{aligned}$$

A simplification of B leads to

$$\begin{aligned}
B\left(V,t,x_t,u\right) &= x^T(t)\left[A_0^T H + HA_0 + \sum_{i=1}^{n} G_i + C_{00}\right]x(t) \\
&\quad + \sum_{i=1}^{n}\left[x^T(t-\tau_i)[A_i^T H + C_{i0}]x(t) + x^T(t)[HA_i + C_{0i}]x(t-\tau_i)\right] \\
&\quad + \sum_{i=1}^{n} x^T(t-\tau_i)[C_{ii} - G_i]x(t-\tau_i) + 2x^T(t)Hcu + du^2. \qquad (3.138)
\end{aligned}$$

Looking for an extremum of (3.138), with respect to u, we get

$$B'_u(V,t,x_t,u) = 2c^T Hx(t) + 2du = 0,$$

that is,

$$u = -\frac{1}{d}c^T Hx(t), \qquad (3.139)$$

which is the minimum of the function B because $B''_{uu} = 2d > 0$. Since

$$\begin{aligned}
&2x^T(t)Hcu + du^2 \\
&\quad = -\frac{2}{d}x^T(t)Hcc^T Hx(t) + \frac{1}{d}x^T(t)Hcc^T Hx(t) = -\frac{1}{d}x^T(t)Hcc^T Hx(t),
\end{aligned}$$

for (3.12) to hold, that is, for

$$B\left(V,t,x_t,-\frac{1}{d}c^T Hx(t)\right) \equiv 0$$

we need

$$\begin{aligned}
x^T(t)&\left[A_0^T H + HA_0 + \sum_{i=1}^n G_i + C_{00} - \frac{1}{d} Hcc^T H\right] x(t) \\
&+ \sum_{i=1}^n \left[x^T(t-\tau_i)[A_i^T H + C_{i0}]x(t) + x^T(t)[HA_i + C_{0i}]x(t-\tau_i)\right] \\
&+ \sum_{i=1}^n x^T(t-\tau_i)[C_{ii} - G_i]x(t-\tau_i) \equiv 0. \qquad (3.140)
\end{aligned}$$

The identity (3.140) will hold if

$$\begin{aligned}
A_0^T H + HA_0 + C_{00} + \sum_{i=1}^n G_i - \frac{1}{d} Hcc^T H &= \Theta_{m\times m}, \\
A_i^T H + C_{i0} &= \Theta_{m\times m}, \ i = 1, \dots, n \\
C_{ii} &= G_i, \ i = 1, \dots, n
\end{aligned}$$

that is, if the assumptions (3.135), (3.136) hold and in (3.134) $G_i = C_{ii}$, $i = 1, \dots, n$. Thus, for the control function defined by (3.139), that is, for u_0 defined by (3.137) the system (3.131) is asymptotically stable and the quality criterion (3.132) takes a minimum value. □

Example 3.4.14. Consider the system (3.131) with $n = m = r = 2$ and

$$A_0 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \ A_1 = \begin{pmatrix} -1 & -0.25 \\ -0.5 & -1 \end{pmatrix}, \ A_2 = \begin{pmatrix} -1 & -0.2 \\ -0.1 & -1 \end{pmatrix}, \ c = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

i.e.,

$$\begin{aligned}
x_1'(t) =& -2x_1(t) + x_2(t) - x_1(t-\tau) - 0.25x_2(t-\tau) - x_1(t-\delta) \\
&- 0.2x_2(t-\delta) + u(t), \\
x_2'(t) =& \ x_1(t) - 2x_2(t) - 0.5x_1(t-\tau) - x_2(t-\tau) - 0.1x_1(t-\delta) \\
&- x_2(t-\delta) + u(t) \qquad (3.141)
\end{aligned}$$

with the quadratic quality criterion (3.132), where

$$C_{11} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \ C_{12} = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}, \ C_{21} = \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix}, \ C_{22} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix},$$

$$C_{13} = \begin{pmatrix} c_1^* & c_2^* \\ c_3^* & c_4^* \end{pmatrix}, \ C_{31} = \begin{pmatrix} c_1^* & c_3^* \\ c_2^* & c_4^* \end{pmatrix}, \ C_{33} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \ d = 1$$

i.e.,

$\omega(t, x_t, u)$

$$
\begin{aligned}
&= \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}^T \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}^T \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \begin{pmatrix} x_1(t-\tau_1) \\ x_2(t-\tau_1) \end{pmatrix} \\
&\quad + \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}^T \begin{pmatrix} c_1^* & c_2^* \\ c_3^* & c_4^* \end{pmatrix} \begin{pmatrix} x_1(t-\tau_2) \\ x_2(t-\tau_2) \end{pmatrix} + \begin{pmatrix} x_1(t-\tau_1) \\ x_2(t-\tau_1) \end{pmatrix}^T \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \\
&\quad + \begin{pmatrix} x_1(t-\tau_2) \\ x_2(t-\tau_2) \end{pmatrix}^T \begin{pmatrix} c_1^* & c_3^* \\ c_2^* & c_4^* \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} x_1(t-\tau_1) \\ x_2(t-\tau_1) \end{pmatrix}^T \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1(t-\tau_1) \\ x_2(t-\tau_1) \end{pmatrix} \\
&\quad + \begin{pmatrix} x_1(t-\tau_2) \\ x_2(t-\tau_2) \end{pmatrix}^T \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1(t-\tau_2) \\ x_2(t-\tau_2) \end{pmatrix} + u^2 \\
&= 3x_1^2(t) + 3x_2^2(t) + 2c_1x_1(t)x_1(t-\tau_1) + 2c_3x_1(t-\tau_1)x_2(t) \\
&\quad + 2c_2x_1(t)x_2(t-\tau_1) + 2c_4x_2(t)x_2(t-\tau_1) + 2c_1^*x_1(t)x_1(t-\tau_2) \\
&\quad + 2c_3^*x_1(t-\tau_2)x_2(t) + 2c_2^*x_1(t)x_2(t-\tau_2) + 2c_4^*x_2(t)x_2(t-\tau_2) \\
&\quad + 3x_1^2(t-\tau_1) + 3x_2^2(t-\tau_1) + 3x_1^2(t-\tau_2) + 3x_2^2(t-\tau_2) + u^2.
\end{aligned}
$$

By formula (3.137), we look for the optimal stabilization control function in the form

$$
u_0 = -\frac{1}{d}c^T Hx(t) = -\begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -(h_1+h_2)x_1 - (h_2+h_3)x_2. \tag{3.142}
$$

Let us determine the matrix H. In our case, we can compute the expression (3.135), i.e.,

$$
\begin{aligned}
&A_0^T H + HA_0 + C_{11} + C_{22} + C_{33} - \frac{1}{d} Hcc^T H \\
&= \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}^T \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix} + \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \\
&\quad + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} h_1 & h_2 \\ h_2 & h_3 \end{pmatrix} = \Theta_{2\times 2}
\end{aligned}
$$

which means that

$$
\begin{cases}
-4h_1 + 2h_2 + 9 - (h_1+h_2)^2 = 0, \\
h_1 - 4h_2 + h_3 - (h_1+h_2)(h_2+h_3) = 0, \\
2h_2 - 4h_3 + 9 - (h_2+h_3)^2 = 0.
\end{cases}
$$

By the "WolframAlpha" software, we obtain two sets of solutions of this system

$$
\begin{aligned}
h_1 = h_3 = \frac{1}{2} + \frac{\sqrt{19}}{4}, \ h_2 = -1 + \frac{\sqrt{19}}{4}, \\
h_1 = h_3 = \frac{1}{2} - \frac{\sqrt{19}}{4}, \ h_2 = -1 - \frac{\sqrt{19}}{4}.
\end{aligned}
$$

Only the first solution is suitable for the matrix H to be positive-definite. Therefore,

$$H = \begin{pmatrix} \dfrac{1}{2} + \dfrac{\sqrt{19}}{4} & -1 + \dfrac{\sqrt{19}}{4} \\ -1 + \dfrac{\sqrt{19}}{4} & \dfrac{1}{2} + \dfrac{\sqrt{19}}{4} \end{pmatrix}.$$

As (3.136) should hold as well, we obtain

$$C_{01} = C_{10}^T = -HA_1 = \begin{pmatrix} 1.63459 & 0.487156 \\ 0.884587 & 1.61216 \end{pmatrix},$$

$$C_{02} = C_{20}^T = -HA_2 = \begin{pmatrix} 1.5987 & 0.40767 \\ 0.248697 & 1.60767 \end{pmatrix},$$

which is sufficient for (3.133) to be a positive-definite matrix. In our case, we have

$$\mathcal{C} = \begin{pmatrix} C_{00} & C_{01} & C_{02} \\ C_{10} & C_{11} & C_{12} \\ C_{20} & C_{21} & C_{22} \end{pmatrix},$$

where C_{12}, C_{21} are null matrices, that is,

$$\mathcal{C} = \begin{pmatrix} 3 & 0 & 1.63459 & 0.487156 & 1.5987 & 0.40767 \\ 0 & 3 & 0.884587 & 1.61216 & 0.248697 & 1.60767 \\ 1.63459 & 0.884587 & 3 & 0 & 0 & 0 \\ 0.487156 & 1.61216 & 0 & 3 & 0 & 0 \\ 1.5987 & 0.248697 & 0 & 0 & 3 & 0 \\ 0.40767 & 1.60767 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

By (3.142) the optimal stabilization control function equals

$$u_0 = \frac{1 - \sqrt{19}}{2}(x_1(t) + x_2(t))$$

with the system (3.141) taking the form

$$\begin{aligned} x_1'(t) = &- \frac{3 + \sqrt{19}}{2} x_1(t) + \frac{3 - \sqrt{19}}{2} x_2(t) - \quad x_1(t-\tau) - 0.25 x_2(t-\tau) \\ &- \quad x_1(t-\delta) - 0.2 x_2(t-\delta), \\ x_2'(t) = &\ \frac{3 - \sqrt{19}}{2} x_1(t) - \frac{3 + \sqrt{19}}{2} x_2(t) - 0.5 x_1(t-\tau) - \quad x_2(t-\tau) \\ &- 0.1 x_1(t-\delta) - \quad x_2(t-\delta). \end{aligned}$$

3.4.6 Systems with multiple delays and a control vector-function

In this part, we consider systems of linear differential equations with delays

$$x'(t) = \sum_{i=0}^{n} A_i x(t-\tau_i) + Cu, \ \ t \geq t_0, \tag{3.143}$$

where A_i, $i = 0, \dots, n$ are $m \times m$ real matrices, C is an $m \times r$ real matrix, $0 = \tau_0 < \tau_1 < \dots < \tau_n$, $x\colon [t_0 - \tau, \infty) \to \mathbb{R}^m$, $t_0 \in \mathbb{R}$ and $u\colon \mathcal{D}_1 \to \mathbb{R}^r$ is a control function. Set $\tau := \tau_n$. A minimization problem

$$I = \min_u \int_{t_0}^{\infty} \omega\left(t, x_t, u\right) \mathrm{d}t, \tag{3.144}$$

where

$$\begin{aligned}\omega\left(t, x_t, u\right) := \sum_{i=0}^{n} x^T(t-\tau_i) C_{ii} x(t-\tau_i) &+ \sum_{i=1}^{n} x^T(t) C_{0i} x(t-\tau_i) \\ + \sum_{i=1}^{n} x^T(t-\tau_i) C_{i0} x(t) &+ \sum_{i=0}^{n} u^T D_i x(t-\tau_i) \\ &+ \sum_{i=0}^{n} x^T(t-\tau_i) D_i^* u + u^T D u \end{aligned} \tag{3.145}$$

will be solved for the system (3.143), where C_{ii} are $m \times m$ constant symmetric matrices, C_{0i} and C_{i0}, $C_{0i} = C_{i0}^T$ are $m \times m$ constant matrices, D is an $r \times r$ symmetric matrix and D_i, D_i^*, $D_i = (D_i^*)^T$ are $r \times m$ and $m \times r$ constant matrices, respectively. Define auxiliary matrices $C_{ij} = C_{ji} = \Theta_{m \times m}$, $(i > j \geq 1,\ i, j = 1, \dots, n)$. Let $X(t)$ be an $[(n+1)m + r] \times 1$ vector defined by the formula

$$X(t) = (x^T(t), x^T(t-\tau_1), \dots, x^T(t-\tau_n), u)^T$$

and

$$\mathcal{C} = \begin{pmatrix} C_{00} & C_{01} & \dots & C_{0n} & D_0^* \\ C_{10} & C_{11} & \dots & C_{1n} & D_1^* \\ \vdots & \vdots & \ddots & \vdots & \\ C_{n0} & C_{n1} & \dots & C_{nn} & D_n^* \\ D_0 & D_1 & \dots & D_n & D \end{pmatrix}. \tag{3.146}$$

Then, the formula (3.145) can be written in the form

$$\omega\left(t, x_t, u\right) = X^T(t) \mathcal{C} X(t).$$

Below we assume that the matrix $\mathcal{C}$ is positive-definite, that is, the functional $\omega(t, x_t, u)$ is positive-definite. In the following, we will employ a Lyapunov-Krasovskii functional (3.134), that is

$$V(t, x_t) = x^T(t) H x(t) + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} x^T(s) G_i x(s) \mathrm{d}s,$$

where $m \times m$ matrices H and G_i, $i = 1, \dots, n$ are constant, positive-definite and symmetric. Their elements will be defined in the formulation of the theorem below. In the proof, we use some well-known formulas for vectors and matrices, taking into account their assumed properties, such us $(A_0 x(t))^T = x^T(t) A_0^T$, $(D^{-1})^T = D^{-1}$, etc., without mentioning this in each particular case (for matrix calculus we refer, for example, to [32]). Matrix computations are performed in detail.

Theorem 3.4.15. *Assume that the matrix $\mathcal{C}$ is positive-definite and there exist positive-definite symmetric matrices H and G_i, $i = 1, \dots, n$, satisfying*

$$A_0^T H + H A_0 + C_{00} + \sum_{i=1}^{n} G_i - [HC + D_0^*] D^{-1} \Big[C^T H + D_0 \Big] = \Theta_{m \times m}, \tag{3.147}$$

$$A_i^T H + C_{i0} - D_i^* D^{-1} \Big[C^T H + D_0 \Big] = \Theta_{m \times m}, \quad i = 1, \dots, n, \tag{3.148}$$

$$G_i - C_{ii} - D_i^* D^{-1} D_i = \Theta_{m \times m}, \quad i = 1, \dots, n. \tag{3.149}$$

If, moreover,

$$D_i^* D^{-1} D_j = \Theta_{m \times m}, \ i, j = 1, \dots, n, \ i \neq j, \tag{3.150}$$

then the optimal stabilization control function of the problem (3.143)–(3.145) *exists and equals*

$$u_0 = -D^{-1} \Big[C^T H + D_0 \Big] x(t) - D^{-1} \sum_{i=1}^{n} D_i x(t - \tau_i). \tag{3.151}$$

Proof. In accordance with Theorem 3.2.1, analyzing the expression B given by (3.11), we get

$$\begin{aligned}
B(V, t, x_t, u) &= \frac{\mathrm{d}V(t, x_t)}{\mathrm{d}t} + \omega(t, x_t, u) \\
&= \left[\sum_{i=0}^{n} A_i x(t - \tau_i) + Cu \right]^T H x(t) + x^T(t) H \left[\sum_{i=0}^{n} A_i x(t - \tau_i) + Cu \right] \\
&\quad + \sum_{i=1}^{n} \Big[x^T(t) G_i x(t) - x^T(t - \tau_i) G_i x(t - \tau_i) \Big] + \sum_{i=0}^{n} x^T(t - \tau_i) C_{ii} x(t - \tau_i) \\
&\quad + \sum_{i=1}^{n} x^T(t) C_{0i} x(t - \tau_i) + \sum_{i=1}^{n} x^T(t - \tau_i) C_{i0} x(t) \\
&\quad + \sum_{i=0}^{n} u^T D_i x(t - \tau_i) + \sum_{i=0}^{n} x^T(t - \tau_i) D_i^* u + u^T D u \\
&= \left[A_0 x(t) + \sum_{i=1}^{n} A_i x(t - \tau_i) + Cu \right]^T H x(t) \\
&\quad + x^T(t) H \left[A_0 x(t) + \sum_{i=1}^{n} A_i x(t - \tau_i) + Cu \right]
\end{aligned}$$

$$+\sum_{i=1}^{n}\Big[x^T(t)G_ix(t)-x^T(t-\tau_i)G_ix(t-\tau_i)\Big]+x^T(t)C_{00}x(t)$$
$$+\sum_{i=1}^{n}x^T(t-\tau_i)C_{ii}x(t-\tau_i)+\sum_{i=1}^{n}x^T(t)C_{0i}x(t-\tau_i)+\sum_{i=1}^{n}x^T(t-\tau_i)C_{i0}x(t)$$
$$+u^TD_0x(t)+x^T(t)D_0^*u+\sum_{i=1}^{n}u^TD_ix(t-\tau_i)$$
$$+\sum_{i=1}^{n}x^T(t-\tau_i)D_i^*u+u^TDu.$$

A simplification of B leads to

$$\begin{aligned}
B\left(V,t,x_t,u\right)=x^T(t)&\left[A_0^TH+HA_0+\sum_{i=1}^{n}G_i+C_{00}\right]x(t)\\
&+\sum_{i=1}^{n}\Big[x^T(t-\tau_i)[A_i^TH+C_{i0}]x(t)+x^T(t)[HA_i+C_{0i}]x(t-\tau_i)\Big]\\
&+\sum_{i=1}^{n}x^T(t-\tau_i)[C_{ii}-G_i]x(t-\tau_i)+u^T(t)\Big[C^TH+D_0\Big]x(t)\\
&+x^T(t)\Big[HC+D_0^*\Big]u+\sum_{i=1}^{n}u^TD_ix(t-\tau_i)+\sum_{i=1}^{n}x^T(t-\tau_i)D_i^*u+u^TDu.
\end{aligned}\tag{3.152}$$

Looking for an extremum of (3.152), we solve the equation

$$B'_u(V,t,x_t,u)=0,$$

where

$$\begin{aligned}
&B'_u(V,t,x_t,u)\\
&=\Bigg[u^T\left[C^TH+D_0\right]x(t)+x^T(t)\left[HC+D_0^*\right]u+\sum_{i=1}^{n}u^TD_ix(t-\tau_i)\\
&\qquad+\sum_{i=1}^{n}x^T(t-\tau_i)D_i^*u+u^TDu\Bigg]'_u\\
&=2\left[C^TH+D_0\right]x(t)+2\sum_{i=1}^{n}D_ix(t-\tau_i)+2Du.
\end{aligned}$$

That is,

$$u=u_0=-D^{-1}\Big[C^TH+D_0\Big]x(t)-D^{-1}\sum_{i=1}^{n}D_ix(t-\tau_i),\tag{3.153}$$

which is the minimum of the function B because $B''_{uu}=2D$ and the matrix D is positive-definite due to the positive-definiteness of $\mathcal{C}$.

Below we prove that u_0 given by (3.153) is the optimal stabilization control function of the problem (3.143)–(3.145), so the formula (3.151) will be proved. Consider the identity (3.12) and derive conditions for its validity, formulated in the theorem.

First, simplify the terms in $B(V, t, x_t, u)$ involving argument u (above, these terms are contained between square brackets in the computation of $B'_u(V, t, x_t, u)$). Using $u = u_0$ defined by (3.153) we get

$$
\begin{aligned}
& u_0^T\Big[C^T H + D_0\Big]x(t) + x^T(t)\Big[HC + D_0^*\Big]u_0 + \sum_{i=1}^{n} u_0^T D_i x(t-\tau_i) \\
&\quad + \sum_{i=1}^{n} x^T(t-\tau_i) D_i^* u_0 + u_0^T D u_0 \\
={}& - \left(D^{-1}\Big[C^T H + D_0\Big]x(t) + D^{-1}\sum_{j=1}^{n} D_j x(t-\tau_j)\right)^T \Big[C^T H + D_0\Big]x(t) \\
&\quad - x^T(t)\Big[HC + D_0^*\Big]\left(D^{-1}\Big[C^T H + D_0\Big]x(t) + D^{-1}\sum_{j=1}^{n} D_j x(t-\tau_j)\right) \\
&\quad - \sum_{i=1}^{n}\left(D^{-1}\Big[C^T H + D_0\Big]x(t) + D^{-1}\sum_{j=1}^{n} D_j x(t-\tau_j)\right)^T D_i x(t-\tau_i) \\
&\quad - \sum_{i=1}^{n} x^T(t-\tau_i) D_i^* \left(D^{-1}\Big[C^T H + D_0\Big]x(t) + D^{-1}\sum_{j=1}^{n} D_j x(t-\tau_j)\right) \\
&\quad + \left(D^{-1}\Big[C^T H + D_0\Big]x(t) + D^{-1}\sum_{j=1}^{n} D_j x(t-\tau_j)\right)^T \\
&\qquad \times D\left(D^{-1}\Big[C^T H + D_0\Big]x(t) + D^{-1}\sum_{k=1}^{n} D_k x(t-\tau_k)\right) \\
={}& \underbrace{-x^T(t)\left(D^{-1}\Big[C^T H + D_0\Big]\right)^T \Big[C^T H + D_0\Big]x(t)}_{\langle 1\rangle} \\
&\quad \underbrace{-\sum_{j=1}^{n} x^T(t-\tau_j)\left(D^{-1}D_j\right)^T \Big[C^T H + D_0\Big]x(t)}_{\langle 2\rangle} \\
&\quad \underbrace{-x^T(t)\Big[HC + D_0^*\Big]D^{-1}\Big[C^T H + D_0\Big]x(t)}_{\langle 3\rangle} \underbrace{-x^T(t)\Big[HC + D_0^*\Big]D^{-1}\sum_{j=1}^{n} D_j x(t-\tau_j)}_{\langle 4\rangle} \\
&\quad \underbrace{-\sum_{i=1}^{n} x^T(t)\left(D^{-1}\Big[C^T H + D_0\Big]\right)^T D_i x(t-\tau_i)}_{\langle 5\rangle} \\
&\quad \underbrace{-\sum_{i=1}^{n}\left(\sum_{j=1}^{n} x^T(t-\tau_j)\left(D^{-1}D_j\right)^T\right) D_i x(t-\tau_i)}_{\langle 6\rangle}
\end{aligned}
$$

$$\underbrace{-\sum_{i=1}^{n} x^T(t-\tau_i)D_i^* D^{-1}\Big[C^T H + D_0\Big]x(t)}_{\langle 7\rangle}\underbrace{-\sum_{i=1}^{n} x^T(t-\tau_i)D_i^* D^{-1}\sum_{j=1}^{n} D_j x(t-\tau_j)}_{\langle 8\rangle}$$

$$\underbrace{+x^T(t)\left(D^{-1}\Big[C^T H + D_0\Big]\right)^T\Big[C^T H + D_0\Big]x(t)}_{\langle 9\rangle}$$

$$\underbrace{+x^T(t)\left(D^{-1}\Big[C^T H + D_0\Big]\right)^T\sum_{k=1}^{n} D_k x(t-\tau_k)}_{\langle 10\rangle}$$

$$\underbrace{+\sum_{j=1}^{n} x^T(t-\tau_j)\left(D^{-1}D_j\right)^T\Big[C^T H + D_0\Big]x(t)}_{\langle 11\rangle}$$

$$\underbrace{+\sum_{j=1}^{n} x^T(t-\tau_j)\left(D^{-1}D_j\right)^T\sum_{k=1}^{n} D_k x(t-\tau_k)}_{\langle 12\rangle}.$$

Finally, the following simplification is carried out: expressions $\langle 1\rangle$, $\langle 2\rangle$, $\langle 5\rangle$ and $\langle 8\rangle$ are rewritten, as indicated. The sum of $\langle 3\rangle$ and $\langle 9\rangle$ equals zero, as well as the sum of $\langle 4\rangle$ and $\langle 10\rangle$, $\langle 6\rangle$ and $\langle 12\rangle$, $\langle 7\rangle$ and $\langle 11\rangle$. We have

$$\begin{aligned}
&u_0^T\Big[C^T H + D_0\Big]x(t) + x^T(t)\Big[HC + D_0^*\Big]u_0 + \sum_{i=1}^{n} u_0^T D_i x(t-\tau_i)\\
&= \underbrace{-x^T(t)\Big[HC + D_0^*\Big]D^{-1}\Big[C^T H + D_0\Big]x(t)}_{\langle 1\rangle}\underbrace{-\sum_{j=1}^{n} x^T(t)\Big[HC + D_0^*\Big]D^{-1}D_j x(t-\tau_j)}_{\langle 5\rangle}\\
&\quad \underbrace{-\sum_{i=1}^{n} x^T(t-\tau_i)D_i^* D^{-1}\Big[C^T H + D_0\Big]x(t)}_{\langle 2\rangle}\underbrace{-\sum_{i=1}^{n} x^T(t-\tau_i)D_i^* D^{-1}\sum_{j=1}^{n} D_j x(t-\tau_j)}_{\langle 8\rangle}.
\end{aligned} \tag{3.154}$$

For (3.12) to hold we need to transform (3.152) using (3.154) to derive

$$\begin{aligned}
&B\left(V, t, x_t, -D^{-1}\Big[C^T H + D_0\Big]x(t) - D^{-1}\sum_{i=1}^{n} D_i x(t-\tau_i)\right)\\
&= x^T(t)\left[A_0^T H + HA_0 + \sum_{i=1}^{n} G_i + C_{00}\right]x(t)\\
&\quad + \sum_{i=1}^{n}\Big[x^T(t-\tau_i)[A_i^T H + C_{i0}]x(t) + x^T(t)[HA_i + C_{0i}]x(t-\tau_i)\Big]\\
&\quad + \sum_{i=1}^{n} x^T(t-\tau_i)[C_{ii} - G_i]x(t-\tau_i)\\
&\quad - x^T(t)\Big[HC + D_0^*\Big]D^{-1}\Big[C^T H + D_0\Big]x(t) - \sum_{j=1}^{n} x^T(t)\Big[HC + D_0^*\Big]D^{-1}D_j x(t-\tau_j)
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=1}^{n} x^T(t-\tau_i) D_i^* D^{-1}\Big[C^T H + D_0\Big] x(t) - \sum_{i=1}^{n} x^T(t-\tau_i) D_i^* D^{-1} \sum_{j=1}^{n} D_j x(t-\tau_j) \\
=& x^T(t)\left[A_0^T H + H A_0 + \sum_{i=1}^{n} G_i + C_{00} - \Big[HC + D_0^*\Big] D^{-1}\Big[C^T H + D_0\Big]\right] x(t) \\
&+ 2\sum_{i=1}^{n} x^T(t-\tau_i)[A_i^T H + C_{i0} - D_i^* D^{-1}\Big[C^T H + D_0\Big]] x(t) \\
&+ \sum_{i=1}^{n} x^T(t-\tau_i)[C_{ii} - G_i - D_i^* D^{-1} D_i] x(t-\tau_i) \\
&- \sum_{i=1}^{n} x^T(t-\tau_i) D_i^* D^{-1} \sum_{j=1, j\neq i}^{n} D_j x(t-\tau_j) \equiv 0.
\end{aligned} \tag{3.155}
$$

The identity (3.155) will hold if

$$
\begin{aligned}
&A_0^T H + H A_0 + C_{00} + \sum_{i=1}^{n} G_i - \Big[HC + D_0^*\Big] D^{-1}\Big[C^T H + D_0\Big] = \Theta_{m\times m}, \\
&A_i^T H + C_{i0} - D_i^* D^{-1}\Big[C^T H + D_0\Big] = \Theta_{m\times m}, \quad i = 1, \dots, n, \\
&C_{ii} - G_i - D_i^* D^{-1} D_i = \Theta_{m\times m}, \quad i = 1, \dots, n, \\
&D_i^* D^{-1} D_j = \Theta_{m\times m}, \quad i, j = 1, \dots, n,\ j \neq i,
\end{aligned}
$$

that is if the assumptions (3.147)–(3.150) are fulfilled. All the assumptions of Theorem 3.2.1 are fulfilled, therefore, for the control function defined by (3.151) and the Lyapunov-Krasovskii functional (3.134), the system (3.143) is asymptotically stable and the quality criterion (3.144) takes a minimum value. □

As a particular case of Theorem 3.4.15, consider the system (3.143) with the quality criterion (3.144) where matrices D_i, D_i^*, $i = 0, \dots, n$ are zero matrices, that is, let

$$
\begin{aligned}
\omega(t, x_t, u) := \sum_{i=0}^{n} x^T(t-\tau_i) C_{ii} x(t-\tau_i) + \sum_{i=1}^{n} x^T(t) C_{0i} x(t-\tau_i) \\
+ \sum_{i=1}^{n} x^T(t-\tau_i) C_{i0} x(t) + u^T D u.
\end{aligned} \tag{3.156}
$$

Then, the following holds.

Theorem 3.4.16. *Assume that the matrix $\mathcal{C}$ is positive-definite and there exist positive-definite symmetric matrices H and G_i, $i = 1, \dots, n$, satisfying*

$$
A_0^T H + H A_0 + C_{00} + \sum_{i=1}^{n} G_i - H C D^{-1} C^T H = \Theta_{m\times m}, \tag{3.157}
$$

$$
A_i^T H + C_{i0} = \Theta_{m\times m}, \ \ i = 1, \dots, n. \tag{3.158}
$$

If, moreover

$$
G_i = C_{ii}, \ \ i = 1, \dots, n, \tag{3.159}
$$

then the optimal stabilization control function of the problem (3.143), (3.144), (3.156) *exists and equals*

$$u_0 = -D^{-1}C^T Hx(t). \tag{3.160}$$

Example 3.4.17. Consider the system (3.143) with $n = m = r = 2$ and

$$A_0 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1 & -0.1 \\ -0.5 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & -0.2 \\ -0.1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix},$$

that is,

$$\begin{aligned} x_1'(t) = &- 2x_1(t) + x_2(t) - \quad x_1(t-\tau_1) - 0.1x_2(t-\tau_1) - \quad x_1(t-\tau_2) \\ &- 0.2x_2(t-\tau_2) + \ u_1(t) + u_2(t), \\ x_2'(t) = &\quad x_1(t) - 2x_2(t) - 0.5x_1(t-\tau_1) - \quad x_2(t-\tau_1) - 0.1x_1(t-\tau_2) \\ &- \quad x_2(t-\tau_2) + 2u_1(t) + u_2(t). \end{aligned} \tag{3.161}$$

Let the matrices in (3.156) be defined as follows

$$C_{00} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \quad C_{01} = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}, \quad C_{10} = \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix}, \quad C_{11} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix},$$

$$C_{02} = \begin{pmatrix} c_1^* & c_2^* \\ c_3^* & c_4^* \end{pmatrix}, \quad C_{20} = \begin{pmatrix} c_1^* & c_3^* \\ c_2^* & c_4^* \end{pmatrix}, \quad C_{22} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then,

$$\begin{aligned} &\omega\,(t, x_t, u) \\ &= \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}^T \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}^T \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \begin{pmatrix} x_1(t-\tau_1) \\ x_2(t-\tau_1) \end{pmatrix} \\ &\quad + \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}^T \begin{pmatrix} c_1^* & c_2^* \\ c_3^* & c_4^* \end{pmatrix} \begin{pmatrix} x_1(t-\tau_2) \\ x_2(t-\tau_2) \end{pmatrix} + \begin{pmatrix} x_1(t-\tau_1) \\ x_2(t-\tau_1) \end{pmatrix}^T \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \\ &\quad + \begin{pmatrix} x_1(t-\tau_2) \\ x_2(t-\tau_2) \end{pmatrix}^T \begin{pmatrix} c_1^* & c_3^* \\ c_2^* & c_4^* \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} x_1(t-\tau_1) \\ x_2(t-\tau_1) \end{pmatrix}^T \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1(t-\tau_1) \\ x_2(t-\tau_1) \end{pmatrix} \\ &\quad + \begin{pmatrix} x_1(t-\tau_2) \\ x_2(t-\tau_2) \end{pmatrix}^T \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1(t-\tau_2) \\ x_2(t-\tau_2) \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ &= 3x_1^2(t) + 3x_2^2(t) + 2c_1x_1(t)x_1(t-\tau_1) + 2c_3x_1(t-\tau_1)x_2(t) \\ &\quad + 2c_2x_1(t)x_2(t-\tau_1) + 2c_4x_2(t)x_2(t-\tau_1) + 2c_1^*x_1(t)x_1(t-\tau_2) \\ &\quad + 2c_3^*x_1(t-\tau_2)x_2(t) + 2c_2^*x_1(t)x_2(t-\tau_2) + 2c_4^*x_2(t)x_2(t-\tau_2) \\ &\quad + 3x_1^2(t-\tau_1) + 3x_2^2(t-\tau_1) + 3x_1^2(t-\tau_2) + 3x_2^2(t-\tau_2) + u_1^2 + u_2^2. \end{aligned}$$

If it is possible to find a matrix H and the entries c_i, c_i^*, $i=1,2$ of the matrices C_{01}, C_{10}, C_{02} and C_{20} then, by the formula (3.160), we obtain the optimal stabilization control function in the form

$$\begin{aligned}
u_0 &= -D^{-1}C^T Hx(t) = -\begin{pmatrix}1&0\\0&1\end{pmatrix}^{-1}\begin{pmatrix}1&1\\2&1\end{pmatrix}^{T}\begin{pmatrix}h_1&h_2\\h_2&h_3\end{pmatrix}x(t)\\
&= -\begin{pmatrix}1&2\\1&1\end{pmatrix}\begin{pmatrix}h_1&h_2\\h_2&h_3\end{pmatrix}x(t) = \begin{pmatrix}-(h_1+2h_2)&-(h_2+2h_3)\\-(h_1+h_2)&-(h_2+h_3)\end{pmatrix}\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}. \quad (3.162)
\end{aligned}$$

We need to find the matrix H. In our case, we verify the expression (3.157), using (3.159). Set

$$\mathcal{A} := A_0^T H + HA_0 + C_{00} + C_{11} + C_{22} - HCD^{-1}C^T H,$$

where

$$\mathcal{A} = \begin{pmatrix}\mathcal{A}_{11}&\mathcal{A}_{12}\\\mathcal{A}_{21}&\mathcal{A}_{22}\end{pmatrix}.$$

That is,

$$\begin{aligned}
\mathcal{A} :=& \begin{pmatrix}-2&1\\1&-2\end{pmatrix}^{T}\begin{pmatrix}h_1&h_2\\h_2&h_3\end{pmatrix}+\begin{pmatrix}h_1&h_2\\h_2&h_3\end{pmatrix}\begin{pmatrix}-2&1\\1&-2\end{pmatrix}+\begin{pmatrix}3&0\\0&3\end{pmatrix}+\begin{pmatrix}3&0\\0&3\end{pmatrix}+\begin{pmatrix}3&0\\0&3\end{pmatrix}\\
&-\begin{pmatrix}h_1&h_2\\h_2&h_3\end{pmatrix}\begin{pmatrix}1&1\\2&1\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}^{-1}\begin{pmatrix}1&1\\2&1\end{pmatrix}^{T}\begin{pmatrix}h_1&h_2\\h_2&h_3\end{pmatrix}\\
=& \begin{pmatrix}-2h_1+h_2&-2h_2+h_3\\h_1-2h_2&h_2-2h_3\end{pmatrix}+\begin{pmatrix}-2h_1+h_2&h_1-2h_2\\-2h_2+h_3&h_2-2h_3\end{pmatrix}+\begin{pmatrix}9&0\\0&9\end{pmatrix}\\
&-\begin{pmatrix}h_1+2h_2&h_1+h_2\\h_2+2h_3&h_2+h_3\end{pmatrix}\begin{pmatrix}h_1+2h_2&h_2+2h_3\\h_1+h_2&h_2+h_3\end{pmatrix}\\
=& \begin{pmatrix}-4h_1+2h_2+9&h_1-4h_2+h_3\\h_1-4h_2+h_3&2h_2-4h_3+9\end{pmatrix}-
\end{aligned}$$

$$\begin{pmatrix}(h_1+2h_2)^2+(h_1+h_2)^2&(h_1+2h_2)(h_2+2h_3)+(h_1+h_2)(h_2+h_3)\\(h_1+2h_2)(h_2+2h_3)+(h_1+h_2)(h_2+h_3)&(h_2+2h_3)^2+(h_2+h_3)^2\end{pmatrix}.$$

We get

$$\begin{aligned}
\mathcal{A}_{11} &= -4h_1+2h_2+9-(h_1+2h_2)^2-(h_1+h_2)^2,\\
\mathcal{A}_{12} &= h_1-4h_2+h_3-(h_1+2h_2)(h_2+2h_3)-(h_1+h_2)(h_2+h_3),\\
\mathcal{A}_{21} &= \mathcal{A}_{12},\\
\mathcal{A}_{22} &= 2h_2-4h_3+9-(h_2+2h_3)^2-(h_2+h_3)^2.
\end{aligned}$$

Then, $\mathcal{A} = \Theta_{2\times 2}$ if

$$\begin{aligned} -4h_1 + 2h_2 + 9 - (h_1 + 2h_2)^2 - (h_1 + h_2)^2 &= 0, \\ h_1 - 4h_2 + h_3 - (h_1 + 2h_2)(h_2 + 2h_3) - (h_1 + h_2)(h_2 + h_3) &= 0, \\ 2h_2 - 4h_3 + 9 - (h_2 + 2h_3)^2 - (h_2 + h_3)^2 &= 0. \end{aligned}$$

By the "WolframAlpha" software, we obtain four sets of solutions of this system

$$\begin{aligned} &h_1 \doteq 1.45357, && h_2 \doteq -0.178416, && h_3 \doteq 1.04933, \\ &h_1 \doteq 0.703179, && h_2 \doteq -1.26542, && h_3 \doteq -0.525272, \\ &h_1 \doteq -23.7181, && h_2 \doteq 12.4297, && h_3 \doteq -5.26584, \\ &h_1 \doteq -26.7254, && h_2 \doteq 17.267, && h_3 \doteq -13.0469. \end{aligned}$$

Only the first one is suitable for the matrix H to be positive-definite. Therefore, (using the above-mentioned values, the following computations are only approximate, further, we do not mention this circumstance)

$$H = \begin{pmatrix} 1.45357 & -0.178416 \\ -0.178416 & 1.04933 \end{pmatrix}.$$

As (3.158) should hold as well,

$$\begin{aligned} C_{01} = C_{10}^T = -HA_1 &= \begin{pmatrix} 1.36436 & -0.033059 \\ 0.346249 & 1.03149 \end{pmatrix}, \\ C_{02} = C_{20}^T = -HA_2 &= \begin{pmatrix} 1.43573 & 0.112298 \\ -0.073483 & 1.01365 \end{pmatrix}, \end{aligned}$$

which is sufficient for (3.146) to be a positive-definite matrix. In our case, we have

$$\mathcal{C} = \begin{pmatrix} C_{00} & C_{01} & C_{02} & D_0^* \\ C_{10} & C_{11} & C_{12} & D_1^* \\ C_{20} & C_{21} & C_{22} & D_2^* \\ D_0 & D_1 & D_2 & D \end{pmatrix},$$

where matrices C_{12}, C_{21}, D_i, D_i^*, $i = 0, 1, 2$ are null matrices, that is,

$$\mathcal{C} = \begin{pmatrix} 3 & 0 & 1.36436 & -0.033059 & 1.43573 & 0.112298 & 0 & 0 \\ 0 & 3 & 0.346249 & 1.03149 & -0.073483 & 1.01365 & 0 & 0 \\ 1.36436 & 0.346249 & 3 & 0 & 0 & 0 & 0 & 0 \\ -0.033059 & 1.03149 & 0 & 3 & 0 & 0 & 0 & 0 \\ 1.43573 & -0.073483 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0.112298 & 1.01365 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By (3.162), the optimal stabilization control function equals

$$u_0 = \begin{pmatrix} -1.096738 & -1.920244 \\ -1.275154 & -0.870914 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

with the system (3.161) taking the form

$$\begin{aligned} x_1'(t) &= -\,4.371892 x_1(t) - 1.791158 x_2(t) - x_1(t-\tau_1) - 0.1 x_2(t-\tau_1), \\ x_2'(t) &= -\,2.46863\ \ x_1(t) - 6.711402 x_2(t) - 0.5 x_1(t-\tau_1) - x_2(t-\tau_1). \end{aligned}$$

4 CONCLUSION

The thesis considers the problem of optimal stabilization for ordinary and functional differential systems. It is based on the result [Theorem 2.2.1, page 21] given in Malkin's book [45, Theorem IV, page 485]. The book [45] is a revised edition of the book [46] and, furthermore, contains new parts - Additions I–IV, prepared by Malkin's followers led by academician N. Krasovskii. In the thesis, first Theorem 2.2.1 was applied to some classes of linear non-delayed differential equations and then the previous result was extended to delayed differential equations and systems. If the delay vanishes ($\tau = 0$), our results reduce back to those already known from [45].

The main result of the thesis is Theorem 3.2.1 (page 38), which solves the problem of minimizing an integral quality criterion. In order to solve this problem, we find an optimal stabilization control function, which simultaneously guarantees the asymptotic stability of a given system of differential equations. The result obtained is successfully applied to certain classes of linear differential equations with delays. Moreover, the examples demonstrate that the theorem is applicable to nonlinear equations with a delay (Examples 3.3.1–3.3.4, p. 43–46).

The problems and derived results, formulated in the thesis, can serve as a motivation for further research. For example, in the thesis, the assumption *iii*) ($B(V, t, x_t, u_0) = 0$) from Theorem 3.2.1 (page 38) is considered only in cases explicitly solvable with respect to u_0. It is also an open question if the theory of implicit functions can be applied to more complicated cases and, consequently, if the results obtained in the thesis can be extended. Another challenge is to apply the results to linear systems with variable coefficients, first in the case of the coefficients being almost constant (for $t \to \infty$).

As a topic for future research, investigation of the solvability of the matrix equations (in the thesis, for example, equations (3.99), (3.115), (3.135), (3.147)) with respect to the matrix H can be suggested as well.

Application of the main result to linear systems leads to complicated systems of nonlinear equations, which determine the elements of the matrix H that has a crucial role in the formulated criteria. In the examples of this thesis, we sometimes overcome this circumstance by using a suitable software. That is why it could be useful to create a special program for solving certain classes of the problems considered.

BIBLIOGRAPHY

[1] Adimy, M., Crauste, F. Existence, positivity and stability for a nonlinear model of cellular proliferation. *Nonlinear Anal. Real World Appl.*, 6 (2005), no. 2, p. 337–366.

[2] Adimy, M., Pujo-Menjouet, L. A mathematical model describing cellular division with a proliferating phase duration depending on the maturity of cells. *Electron. J. Differential Equations*, 2003, no. 107, 14 pp.

[3] Agarwal, R.P., Berezansky, L., Braverman, E., Domoshnitsky, A. *Nonoscillation Theory of Functional Differential Equations with Applications.* Springer, New York, 2012. 520 pp. ISBN: 978-1-4614-3454-2.

[4] Agarwal, R.P., Bohner, M., Wan-Tong, Li *Nonoscillation and Oscillation: Theory for Functional Differential Equations.* Monographs and Textbooks in Pure and Applied Mathematics, 267. Marcel Dekker, Inc., New York, 2004. 376 pp. ISBN: 0-8247-5845-5.

[5] Alekseev, V.M., Tikhomirov, V.M., Fomin, S.V. *Optimal Control.* (Russian) Moscow, Nauka Publisher, 1979. 432 pp. Translated from the Russian by V. M. Volosov. Contemporary Soviet Mathematics. Consultants Bureau, New York, 1987. 309 pp. ISBN: 0-306-10996-4.

[6] Athans, M., Falb, P.L. *Optimal Control, An Introduction to the Theory and Its Applications.* Dover Publications, Inc., 2007.

[7] Barrea, A., Hernandez, M.E. Optimal control of a delayed breast cancer stem cells nonlinear model. *Optimal Control Appl. Methods*, 37 (2016), no. 2, p. 248–258.

[8] Bekiaris-Liberis, N., Krstic, M. *Nonlinear Control Under Nonconstant Delays.* Advances in Design and Control, 25. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2013. 299 pp. ISBN: 978-1-611973-17-4.

[9] Bellman, R. *Dynamic Programming.* Reprint of the 1957 edition. With a new introduction by Stuart Dreyfus. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 2010. 340 pp. ISBN: 978-0-691-14668-3.

[10] Bokov, G.V. Pontryagin's maximum principle of optimal control problems with time delay. (Russian), *Fundam. Prikl. Mat.*, 15 (2009), no. 5, p. 3–19; translation in *J. Math. Sci. (N.Y.)*, 172 (2011), no. 5, p. 623–634.

[11] Demchenko, H., Diblík, J. A problem of functional minimizing for single delayed differential system. In *Mathematics, Information Technologies and Applied Sciences 2017, Post-conference proceedings of extended versions of selected papers*. Brno: University of Defence, 2017. p. 55-62. ISBN: 978-80-7582-026-6. Available at: `<http://mitav.unob.cz/data/MITAV2017Proceedings.pdf>`.

[12] Demchenko, H., Diblík, J. Optimality conditions for a linear differential system with a single delay. In *MITAV 2017 (Matematika, informační technologie a aplikované vědy)*. Brno: Univerzita obrany v Brně, 2017. p. 1–7. ISBN: 978-80-7231-417-1.

[13] Demchenko, H. Optimality conditions for scalar linear differential system. In *Proceedings of the 23rd Conference STUDENT EEICT 2017*. Brno: Brno University of Technology, Faculty of Elecrical Engineering and Communication, 2017. p. 629–633. ISBN: 978-80-214-5496-5.

[14] Demchenko, H., Diblík, J. Optimality conditions for scalar linear delayed differential equations. In *Proceedings of 16th Conference on Applied Mathematics Aplimat 2017*, First Edition, Bratislava: Slovak University of Technology in Bratislava, 2017, p. 440–444, ISBN: 978-80-227-4650-2. Available at: `<http://toc.proceedings.com/33721webtoc.pdf>`.

[15] Demchenko, H., Diblík, J., Khusainov, D. Optimization of linear differential systems with multiple delay. In *MITAV 2016 (Matematika, informační technologie a aplikované vědy)*. Brno: Univerzita obrany v Brně, 2016. p. 1–7. ISBN: 978-80-7231-464-5.

[16] Demchenko, H. Optimization of linear differential systems with a delay by Lyapunov's direct method. In *Proceedings of the 22nd Conference STUDENT EEICT 2016*. Brno: Brno University of Technology, Faculty of Elecrical Engineering and Communication, 2016. p. 748–752. ISBN: 978-80-214-5350-0.

[17] Demchenko, H., Diblík, J., Khusainov, D. Optimization of linear differential systems with delay by Lyapunov's direct method. In *Mathematics, Information Technologies and Applied Sciences 2015, Post-conference proceedings of extended versions of selected papers*. Brno: University of Defence, 2015. p. 49–57. ISBN: 978-80-7231-436-2. Available at: `<http://mitav.unob.cz/data/MITAV2015Proceedings.pdf>`.

[18] Demchenko, H., Diblík, J., Khusainov, D. Optimization of linear differential systems with delay by Lyapunov's direct method. In *MITAV 2015 (Matematika,*

informační technologie a aplikované vědy). Brno: Univerzita obrany v Brně, 2015. p. 1–6. ISBN: 978-80-7231-998-5.

[19] Demchenko, H. Optimization of linear differential systems by Lyapunov's direct method. In *Proceedings of the 21st Conference STUDENT EEICT 2015.* Brno: Brno University of Technology, Faculty of Elecrical Engineering and Communication, 2015. p. 511–515. ISBN: 978-80-214-5148-3.

[20] Diblík, J. Long-time behaviour of solutions of delayed-type linear differential equations. *Electron. J. Qual. Theory Differ. Equ.*, 2018, no. 47, p. 1–23.

[21] Diblík, J. Asymptotic representation of solutions of equation $\dot{y}(t) = \beta(t)[y(t) - y(t - \tau(t))]$. *J. Math. Anal. Appl.*, 217 (1998), no. 1, p. 200–215.

[22] Diblík, J., Kukharenko, O., Khusainov, D. Solution representation of the first boundary value problem for equations with constant delay. *Bulletin Kiev University, series: Physics and Mathematics*, 1 (2011), p. 59-62.

[23] Driver, R.D. *Ordinary and Delay Differential Equations.* Applied Mathematical Sciences, Vol. 20. Springer-Verlag, New York-Heidelberg, 1977. 501 pp. ISBN: 0-387-90231-7.

[24] Elsgolts, L.E., Norkin, S.B. *Introduction to the Theory of Differential Equations with Delay.* (Russian) Nauka Publisher, Moscow, 1971. 296 pp.

[25] Fel'dbaum, A. A. *Fundamentals of the Theory of Optimal Automatic Systems.* Second revised and enlarged edition. (Russian) Nauka Publisher, Moscow, 1966. 623 pp.

[26] Feng, Y., Tu, D., Li, C., Huang, T. Alternate control delayed systems. *Adv. Difference Equ.* 2015, 2015:146, 12 pp.

[27] Fridman, E. *Introduction to Time-Delay Systems, Analysis and Control.* Systems & Control: Foundations & Applications, Birkhäuser/Springer, Cham, 2014. 362 pp. ISBN: 978-3-319-09392-5; 978-3-319-09393-2.

[28] Fridman, E., Nicaise, S., Valein, J. Stabilization of second order evoluion equations with unbounded feedback with time-dependent delay. *SIAM J. Control Optim.*, 48 (2010), no. 8, p. 5028–5052.

[29] Gabasov, R., Kirillova, F. *The Qualitative Theory of Optimal Processes.* Translated from the Russian by John L. Casti. Control and Systems Theory, Vol. 3. Marcel Dekker, Inc., New York-Basel, 1976. 640 pp.

[30] Gabasov, R., Kirillova, F.M., Prischepova, S.V. *Optimal Feedback Control.* Lecture Notes in Control and Information Sciences, 207. Springer-Verlag London, Ltd., London, 1995. 202 pp. ISBN: 3-540-19991-8.

[31] Gan, Q., Xu, R., Zhang, X., Yang, P. Travelling waves of a three-species Lotka-Volterra food-chain model with spatial diffusion and time delays. *Nonlinear Anal. Real World Appl.*, 11 (2010), no. 4, p. 2817–2832.

[32] Gantmacher, F.R. *The Theory of Matrices*, vol. I, II, AMS Chelsea Publishing, Providence, RI, USA, 2002.

[33] Gyori, I., Ladas, G. *Oscillation Theory of Delay Differential Equations.* With applications. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1991. 368 pp. ISBN: 0-19-853582-1.

[34] Hale, J.K. *Theory of Functional Differential Equations.* Second edition. Applied Mathematical Sciences, Vol. 3. Springer-Verlag, New York-Heidelberg, 1977. 365 pp.

[35] Kim, A.V., Ivanov, A.V. *Systems with Delay.* Analysis, control, and computations. John Wiley & Sons, Inc., Hoboken, NJ; Scrivener Publishing, Salem, MA, 2015. 164 pp. ISBN: 978-1-119-11758-2.

[36] Kirk, D.E. *Optimal Control Theory, An Introduction.* Dover Publications, Inc., 2004.

[37] Krasovskii, N.N. *Stability of Motion.* Applications of Lyapunov's second method to differential systems and equations with delay. Translated by J. L. Brenner Stanford University Press, Stanford, Calif. 1963. 188 pp.

[38] Kukharenko, O., Diblík, J., Morávková, B., Khusainov, D. Delayed exponential functions and their application to representations of solutions of linear equations with constant coefficients and with single delay. In *Proceedings of the 2nd International Conference on Mathematical Models for Engineering Science (MMES '11)*, Puerto De La Cruz, Spain, 2011. p. 82–87.

[39] Laarabi, H., Abta, A., Hattaf, K. Optimal control of a delayed SIRS epidemic model with vaccination and treatment. *Acta Biotheoretica*, 63(2), 2015, p. 87–97.

[40] Letov, A. M. The analytical design of control systems. *Avtomat. i Telemeh*, 22, p. 425–435 (Russian. English summary); translated as *Automat. Remote Control*, 22, 1961, p. 363–372.

[41] Letov, A. M. The analytical design of control systems. V. Further developments in the problem. *Avtomat. i Telemeh*, 23, p. 1405–1413 (Russian. English summary); translated as *Automat. Remote Control*, 23, 1962, p. 1319–1327.

[42] Ling, Q., Jin, X., Huang, Z. Stochastic stability of quasi-integrable Hamiltonian systems with time delay by using Lyapunov function method. *Science China Technological Sciences*, 53(3), 2010, p. 703–712.

[43] Lyapunov, A.M. *Selected Works: Works on the Theory of Stability.* (Russian) Moscow, Nauka Publisher, 2007. 574 pp.

[44] Macki, J., Strauss, A. *Introduction to Optimal Control Theory.* Corrected second printing, Undergraduate Texts in Mathematics, Springer-Verlag, 1995.

[45] Malkin, I.G. *Theory of Stability of Motion.* Second revised edition. (Russian) Moscow, Nauka Publisher, 1966. 530 pp.

[46] Malkin, I.G. *Theory of Stability of Motion.* Translated from a publication of the state publishing house of technical-theoretical literature. Moscow-Leningrad, 1952, United State Atomic Energy Commission, Office of Technical Information, Translation Series, 455 pp. https://babel.hathitrust.org/cgi/pt?id=mdp.39015014359726;view=1up;seq=1.

[47] Maxwell, J.C., Vyshnegradsky, I.A., Stodola, A. *Theory of Automatic Control.* (Russian) Moskow, USSR AS Publisher, 1949. 430 pp.

[48] Meng, Q., Shen, Y. Optimal control for stochastic delay evolution equations. *Appl. Math. Optim.*, 74 (2016), no. 1, p. 53–89.

[49] Metel'skiia, A. V., Urbanb, O. I., Khartovskiib, V. E. Damping of a solution of linear autonomous difference–differential systems with many delays using feedback. Translation of Izv. Ross. Akad. Nauk Teor. Sist. Upr. 2015, no. 2, p. 40–49. *J. Comput. Syst. Sci. Int.*, 54 (2015), no. 2, p. 202–211.

[50] Michiels, W., Niculescu, S.I. *Stability and Stabilization of Time-Delay Systems, An Eigenvalue-Based Approach.* Advances in Design and Control, 12. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2007. 378 pp. ISBN: 978-0-898716-32-0.

[51] Pedregal, P. *Introduction to Optimization.* Texts in Applied Mathematics, 46. Springer-Verlag, New York, 2004. 245 pp. ISBN: 0-387-40398-1.

[52] Pontryagin, L.S., Boltyanskij, V.G., Gamkrelidze, R.V., Mishchenko, E.F. *The Mathematical Theory of Optimal Processes.* (Russian) Moscow, Nauka Publisher, 1983. 392 pp.

[53] Rihan, F.A., Rihan, B.F. Numerical modelling of biological systems with memory using delay differential equations. *Appl. Math. Inf. Sci.*, 9 (2015), no. 3, p. 1645–1658.

[54] Shi, J. Optimal control for stochastic differential delay equations with poisson jumps and applications. *Random Oper. Stoch. Equ.*, 23 (2015), no. 1, p. 39–52.

[55] Yi, S., Nelson, P.W., Ulsoy, A.G. *Analysis and Control Using the Lambert W Function*, World Scientific, 2010.

[56] Volterra, V. *The Mathematical Theory of the Struggle for Existence.* (Russian) Moscow, Nauka Publisher, 1976. 288 pp.

[57] Wang, Q., Chen, F., Huang, F. Maximum principle for optimal control problem of stochastic delay differential equations driven by fractional Brownian motions. *Optimal Control Appl. Methods*, 37 (2016), no. 1, p. 90–107.

[58] Wang, C., Zuo, Z., Lin, Z., Ding, Z. Consensus control of a class of Lipschitz nonlinear systems with input delay. *IEEE Trans. Circuits Syst. I. Regul. Pap.*, 62 (11), 2015, p. 2730–2738.

[59] Wang, H., Zhang, H., Wang, X. Optimal control for stochastic discrete-time systems with multiple input delays. In *Proceedings of the 10th World Congress on Intelligent Control and Automation*, 2012, p.1529–1534.

[60] Zabczyk, J. *Mathematical Control Theory: an Introduction.* Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, 1992. 260 pp. ISBN: 0-8176-3645-5.

[61] Zhu, W., Zhang, Z. Verification theorem of stochastic optimal control with mixed delay and applications to finance. *Asian J. Control*, 17 (2015), no. 4, p. 1285–1295.